

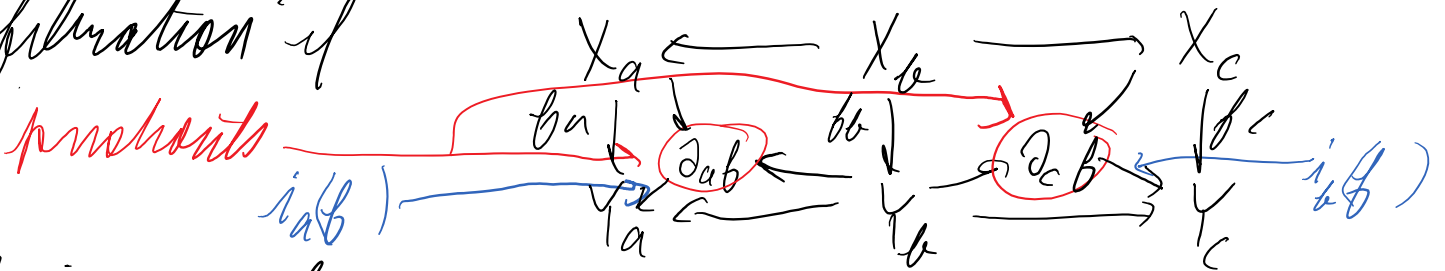
Mc Tague on non-homotopical derived functors

Monday, February 29, 2016 8:41 AM

Ex $J = \{ a \leftarrow b \rightarrow c \}$. $\text{colim} : \text{Top}^J \rightarrow \text{Top}$ is left adjoint of $\Delta : \text{Top} \rightarrow \text{Top}^J$ (constant diagram functor). Consider the case

$$\begin{array}{ccccc}
 X & & D^n & \xleftarrow{S^{n-1}} & D^n & & \text{colim} = S^n \\
 \downarrow f & & \cong \downarrow f_a & & \cong \downarrow f_b & & \\
 Y & & * & \xrightarrow{S^{n-1}} & * & & \text{colim} = *
 \end{array}$$

Colimits do not preserve weak epimorphisms. Top^J has a MC structure in which X is cofibrant but Y is not. $f : X \rightarrow Y$ is a weak epimorphism or fibration if each f_i is. f is a cofibration if



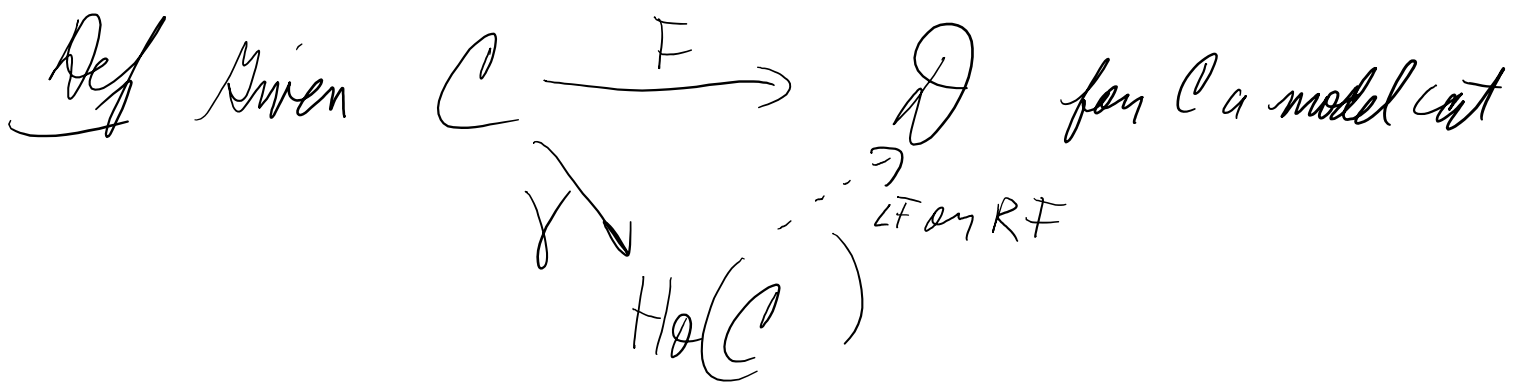
f is a cofibration if f_b and the corner maps i_a and i_b are cofibrations.

Claim X in Top^J is cofibrant if X_a is CW-complex and 2-maps from it are cofibrations.

Ex $* \leftarrow S^{n-1} \rightarrow *$ is not cofibrant

Question When does $F : \mathcal{C} \rightarrow \mathcal{D}$ (functor of model categories) factor thru $\text{Ho}(\mathcal{C})$

categories) factor thru $H_0(\mathcal{C})$



A left (right) derived functor of LF (RF) is a right (left) Kan extension of F along γ .

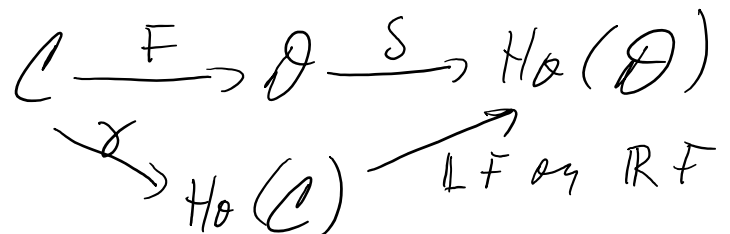
If it exists it comes with nat trans

$$\varepsilon: \text{LF} \circ \gamma \Rightarrow F \quad (\eta: F \Rightarrow \text{RF})$$

Kan extension exists when \mathcal{C} is small and \mathcal{D} is complete / co complete

Prop LF (RF) exists if F converts trivial cofibers (trivial fib) to isomorphisms

Def Given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between model cats



a total left (right) derived functor $\mathbb{L}F$ ($\mathbb{R}F$) is a right (left) Kan extension of SF along γ .

Prop $\mathbb{L}F$ ($\mathbb{R}F$) exists if F converts trivial cofibers (trivial fib) to weak equivs.

Ex For a ring R , the category $Ch_{\mathbb{Z}}$ of nonnegatively graded chain complexes of R -modules has a MC structure in which cofibrant objects are projective chain cxs. and cofibrant replacement of $K(N, 0) : N \leftarrow 0 \leftarrow \dots$ is a projective res. of N .

For a right R -module M , $M \otimes_R (-)$ defines a functor $F : Ch_{\mathbb{Z}} \rightarrow Ch_{\mathbb{Z}}$. It has a $\mathbb{L}F : Ho(Ch_R) \rightarrow Ho(Ch_{\mathbb{Z}})$ and $H_n \mathbb{L}F(K(N, 0)) = Tor_n^R(M, N)$.

Quillen functors

Question: When does an adjunction $F : M \rightleftarrows N : G$ induce an adjunction $\mathbb{L}F : Ho M \rightleftarrows Ho N : \mathbb{R}G$

Def (Quillen pairs)

1) F is a left Quillen functor

2) G is a right Quillen functor

3) (F, U) is a Quillen pair if

a) $F(\text{cofib}) = \text{cofib}$ and $F(\text{triv cofib}) = \text{triv cofib}$

b) $U(\text{fib}) = \text{fib}$ and $U(\text{triv fib}) = \text{triv fib}$

Prop (The conditions above are redundant)

TFAE for $F \dashv U$

i) (F, U) is Quillen pair

ii) a) holds

iii) b) holds

iv) $F(\text{cofib}) = \text{cofib}$ and $U(\text{fib}) = \text{fib}$

v) $F(\text{triv cofib}) = \text{triv cofib}$ and
 $U(\text{triv fib}) = \text{triv fib}$.

vi) and vii) Good behaviour on cofibrant
or fibrant objects.

Prop If (F, U) is a Quillen pair then

a) $F(\text{weak equiv of cofibrant objects}) = \text{weak equiv}$

b) dual to a

c) $\perp F$ and $\perp U$ exist

Thm $\perp F \dashv \perp U$.

Homotopical categories

How much MC theory can we do using only
weak equivs?

Def A homotopical cat is a cat \mathcal{M} equipped
with a wide (contains all objects) subcat \mathcal{W}
whose morphisms (weak equivs) satisfy
the 2 out of 3 property:

$$0 \xrightarrow{f} 0 \xrightarrow{g} 0 \xrightarrow{h} 0.$$

If $g \circ f$ and $h \circ g \in \mathcal{W}$, so are f, g, h and $h \circ f$.

Def. A homotopy functor $F: \mathcal{M} \rightarrow \mathcal{C}$
 (for \mathcal{C} homotopical) is one that takes
 weak eqivs to isos.

A homotopical functor : $F: \mathcal{M} \rightarrow \mathcal{C}$
 (both homotopical) is one that preserves
 weak eqivs.

Note Homotopical cats need not be complete or
 co complete.

Note Any iso in a WE. $X \xrightarrow{b} Y \xrightarrow{b^{-1}} X \xrightarrow{b} Y$
 I_X I_Y

$I_X \in W$ since W is a subcat.

Note. 2 out of 6 \Rightarrow 2 out of 3

$X \xrightarrow{b} Y \xrightarrow{g} Z = Z$, $X \xrightarrow{b} Y = Y \xrightarrow{g} Z$, $X = X \xrightarrow{b} Y \xrightarrow{g} Z$

Prop The weak eqivs in a model cat determine
 a homotopical category

Ex Given a functor $\mathcal{M} \xrightarrow{F} \mathcal{C}$ we can define

$$W = \{ f \in \mathcal{M} \mid F(f) = \text{iso} \}$$

More generally $\mathcal{M} \xrightarrow{F} \mathcal{N} = \text{homotopical cat}$

$$W = \{ f \in \mathcal{M} \mid F(f) = \text{weak eqiv} \}$$

Def A homotopical cat \mathcal{M} has a homotopy
 category $Ho(\mathcal{M})$