McTague on non-homotopical + derived functors

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Ex $J = \{a \leftarrow b \rightarrow c\}$. $\mathrm{colim} : Top^J \to Top$ is left adjoint of $\Delta : Top \to Top^J$ (constant diagram functor). Consider the case

$$\begin{tikzcd} X \arrow[d, "f"'] & D^n \arrow[d, "\simeq f_a"'] & S^{n-1} \arrow[l] \arrow[r] \arrow[d, equal, "\simeq f_b"'] & D^n \arrow[d, "\simeq f_c"'] & \mathrm{colim} = S^n \\ Y & * & S^{n-1} \arrow[l] \arrow[r] & * & \mathrm{colim} = * \end{tikzcd}$$

Colims do not preserve weak equivs. Top^J has a MC structure in which X is cofibrant but Y is not. $f : X \to Y$ is a weak equiv or fibration if each f_j is. f is a cofibration if

$$\begin{tikzcd} X_a \arrow[dr] \arrow[dd, "f_a"'] & & X_b \arrow[ll] \arrow[rr] \arrow[dd, "f_b"] & & X_c \arrow[dl] \arrow[dd, "f_c"] \\ & \partial_a f \arrow[dl, "i_a f"] & & \partial_c f \arrow[dr, "i_b f"'] & \\ Y_a & & Y_b \arrow[ll] \arrow[rr] \arrow[ul] \arrow[ur] & & Y_c \end{tikzcd}$$

pushouts

f is a cofib if f_b and the corner maps i_a and i_b are cofibrations.

Claim X in Top^J is cofibrant if X_b is CW-cx and 2 maps from it are cofibrations.

Ex $* \leftarrow S^{n-1} \rightarrow *$ is not cofibrant

Question When does $F : \mathcal{C} \to \mathcal{D}$ (functor of model categories) factor thru $Ho(\mathcal{C})$

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Def Given
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \\ Ho(\mathcal{C}) \arrow[ur, dashed, "LF \text{ or } RF"'] & \end{tikzcd}$$
for $\mathcal{C}$ a model cat

A left (right) derived functor of LF (RF) is a right (left) Kan extension of F along γ.
If it exists it comes with nat trans
$\varepsilon : LF \circ \gamma \Rightarrow F$ ($\eta : F \Rightarrow RF$)
Kan extension exists when $\mathcal{C}$ is small and $\mathcal{D}$ is complete / cocomplete

Prop LF (RF) exists if F converts trivial cofibs (trivial fibs) to isomorphisms.

Def Given a functor $F : \mathcal{C} \to \mathcal{D}$ between model cats

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "\gamma"'] & \mathcal{D} \arrow[r, "\delta"] & Ho(\mathcal{D}) \\ & Ho(\mathcal{C}) \arrow[ur, "\mathbb{L}F \text{ or } \mathbb{R}F"'] & \end{tikzcd}$$

a total left (right) derived functor $\mathbb{L}F$ ($\mathbb{R}F$) is a right (left) Kan extension of δF along γ.

Prop $\mathbb{L}F$ ($\mathbb{R}F$) exists if F converts trivial cofibs (trivial fibs) to weak equivs.

Ex For a ring R, the category Ch_R of nonnegatively graded chain complexes of R-modules has a MC structure in which cofibrant objects are projective chain cxs. and cofibrant replacement of $K(N,0): N \leftarrow 0 \leftarrow \cdots$ is a projective resol'n of N.

For a right R-module M, $M \otimes_R (-)$ defines a functor $F: Ch_R \to Ch_{\mathbb{Z}}$. It has a $\mathbb{L}F: Ho(Ch_R) \to Ho(Ch_{\mathbb{Z}})$ and

$H_1 \, \mathbb{L}F(K(N,0)) = Tor_1^R(M,N)$.

Quillen functors

Question: When does an adjunction $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$ induce an adjunction $\mathbb{L}F: Ho\,\mathcal{M} \rightleftarrows Ho\,\mathcal{N}: \mathbb{R}G$?

Def (Quillen pairs)

1) F is a left Quillen functor
2) G " right "
3) (F,U) is a Quillen pair if
a) $F(\text{cofib}) = \text{cofib}$ and $F(\text{triv cofib}) = \text{triv cofib}$
b) $U(\text{fib}) = \text{fib}$ and $U(\text{triv fib}) = \text{triv fib}$

Prop (The conditions above are redundant)
TFAE for $F \dashv U$
i) (F,U) is Quillen pair
ii) a) holds.

iii) b) holds

iv) F(cofib) = cofib and U(fib) = fib

v) F(triv cofib) = triv cofib and
U(triv fib) = triv fib.

vi) and vii) Good behavior on cofibrant
on fibrant objects.

<u>Prop</u> If (F, U) is a Quillen pair then

a) F(weak equiv of cofibrant objects) = weak equivs

b) dual to a

c) $\mathbb{L}F$ and $\mathbb{R}U$ exist

Thm $\mathbb{L}F \dashv \mathbb{R}U$.

Homotopical categories

How much MC theory can we do using only weak equivs?

<u>Def</u> A <u>homotopical cat</u> is a cat M equipped with a wide (contains all objects) subcat W whose morphisms (weak equivs) satisfy the 2 out of 6 property;

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet .$$

If gf and $hg \in W$, so are f, g, h and hgf.

Def. A **homotopy functor** $F: \mathcal{M} \to \mathcal{C}$ (for $\mathcal{C}$ homotopical) is one that takes weak equivs to isos.

A **homotopical functor** : $F: \mathcal{M} \to \mathcal{C}$ (both homotopical) is one that preserves weak equivs.

Note Homotopical cats need not be complete or cocomplete.

Note Any iso is a WE.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "1_X"'] & Y \arrow[r, "f^{-1}"] \arrow[rr, bend left, "1_Y"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$1_X \in W$ since W is a subcat.

Note. 2 out of 6 $\Rightarrow$ 2 out of 3

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "fg"] & Y \arrow[r] \arrow[rr, bend right, "g"'] & Z \arrow[r, equal] & Z \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "f"] & Y \arrow[r, equal] \arrow[rr, bend right, "g"'] & Y \arrow[r, "g"] & Z \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r, equal] \arrow[rr, bend left, "f"] & X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Prop The weak equivs in a model cat determine a homotopical category

Ex Given a functor $\mathcal{M} \xrightarrow{F} \mathcal{C}$ we can define

$W = \{ f \in \mathcal{M} \mid F(f) = \text{iso}$.

More generally $\mathcal{M} \xrightarrow{F} \mathcal{N}$ = homotopical cat

$W = \{ f \in \mathcal{M} \mid F(f) = \text{weak equiv} \}$

Def A homotopical cat $\mathcal{M}$ has a homotopy category $Ho(\mathcal{M})$