MZ, QZ, UY, CM, YQ, YZ

# Zeng on model categories

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Two approaches to the homotopy category Ho $\mathcal{M}$ of a model category $\mathcal{M}$.

1. Def Let $\mathcal{C}$ be a category with a subcat $\mathcal{W}$ weak equivalences with the 2 of 3 property. We can form the homotopy "category" Ho $\mathcal{C}$ by inverting all maps in $\mathcal{W}$. There is a potential set theoretic problem in that there may be a proper class of morphisms between 2 objects. We will see that this does <u>not</u> happen when $\mathcal{C}$ is a model category.

There is a functor $\gamma: \mathcal{C} \to Ho(\mathcal{C})$ which is the identity on objects.

<u>Lemma</u> (Universal property of Ho $\mathcal{C}$)

(i) If $F: \mathcal{C} \to \mathcal{D}$ sending weak equivs to isos then $\exists!$ functor $\tilde{F}: Ho(\mathcal{C}) \to \mathcal{D}$ with $\tilde{F}\gamma = F$.

(ii) The above gives an iso of categories between Fun $(Ho\,\mathcal{C}, \mathcal{D})$ and the subcat of Fun$(\mathcal{C}, \mathcal{D})$ of functors sending weak equivs to isos

<u>Prop</u> Let $\mathcal{C}$ be a model category. Let $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ be the full subcats of cofibrant, fibrant and cofibrant fibrant objects. Then

$$\mathrm{Ho}\,\mathcal{C}_{cf} \underset{R}{\overset{}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{C}_c \underset{Q}{\rightleftarrows} \mathrm{Ho}\,\mathcal{C}$$

and $$\mathrm{Ho}\,\mathcal{C}_{cf} \underset{Q}{\rightleftarrows} \mathrm{Ho}\,\mathcal{C}_f \underset{R}{\rightleftarrows} \mathrm{Ho}\,\mathcal{C}$$

are equivalences of categories with inverse functors being replacement functors in red. Note the fibrant replacement of a cofibrant object is also cofibrant (proof omitted).

Approach 2 Homotopy as an equivalence relation.

Def Let $\mathcal{C}$ be a model category with maps $f, g : B \to X$

① A cylinder object is a factorization of the map

$$\begin{tikzcd} B \amalg B \arrow[rr, "\text{fold}"] \arrow[dr, "\text{cofib}"'] & & B \\ & B' \arrow[ur, "\text{trivial fib}"'] & \end{tikzcd}$$

(e.g. the one given by MC5)

② The path object is a factorization

$$\begin{tikzcd} X \arrow[rr, "\Delta"] \arrow[dr, "\text{trivial cofib}"'] & & X \times X \\ & X' \arrow[ur, "\text{fib}"'] & \end{tikzcd}$$

③ A left homotopy from $f$ to $g$ ($f \overset{l}{\sim} g$) is a map $H : B' \to X$ s.t.

$$\begin{tikzcd} B \arrow[dr, "i_0"'] \arrow[drr, bend left, "g"] & & \\ & B' \arrow[r, "H"] & X \\ B \arrow[ur, "i_1"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

④ A <u>right homotopy from $f$ to $g$</u> ($f \overset{r}{\sim} g$) is a map $B \xrightarrow{K} X'$ such that

$$\begin{tikzcd}
 & & X \\
B \arrow[r, "K"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & X' \arrow[ur, "p_0"'] \arrow[dr, "p_1"] & \\
 & & X
\end{tikzcd}$$

⑤ $f$ is <u>homotopic</u> to $g$ ($f \sim g$) if $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$.

⑥ $f$ is a <u>homotopy equivalence</u> if $\exists h: X \to B$ and $h \circ f \sim 1_B$

<u>Thm</u> Assume $B$ is cofibrant and $f, g: B \to X$. Then $f \overset{r}{\sim} g$ iff $f \overset{l}{\sim} g$, homotopy is an equivalence relation, and left/right htpy can be realized by <u>any</u> cylinder/path object. In $C_{cf}$ the homotopy relation is compatible with composition, so we define $C_{cf}/\sim$ and a functor $\delta: C_{cf} \to C_{cf}/\sim$ which is the identity on objects.

<u>Thm</u> A map in $C_{cf}$ is a weak equivalence iff it is a homotopy equivalence.

<u>Cor</u> Let $\gamma: C_{cf} \to Ho(C_{cf})$ and $\delta: C_{cf} \to C_{cf}/\sim$

Cor Let $\gamma : \mathcal{C}_{cf} \longrightarrow Ho(\mathcal{C}_{cf})$ and $\delta : \mathcal{C}_{cf} \longrightarrow \mathcal{C}_{cf}/\sim$ be as above. Then $\exists!$ equiv of categories $j : \mathcal{C}_{cf}/\sim \; \to Ho\,\mathcal{C}_{cf}$

such that $j \circ \gamma = \delta$ and $\delta$ is the identity on objects.

It follows that $\mathrm{Ho}\,\mathcal{C}$ is locally small.

Thm (Fundamental thm of model categories)
Let $\mathcal{C}$ be a model cat, $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$, $Q$ be cofibrant replacement, $R$ be fibrant replacement.

(i) $\mathcal{C}_{cf} \to \mathcal{C}$ gives equivalences
$$\mathcal{C}_{cf}/\sim \xrightarrow{\simeq} \mathrm{Ho}\,\mathcal{C}_{cf} \xrightarrow{\simeq} \mathrm{Ho}\,\mathcal{C}$$

(ii) There are natural isomorphisms
$$\mathcal{C}(QRX, QRY)/\sim \;\cong \mathrm{Ho}\,\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(RQX, RQY)$$

(iii) $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ identifies both left + right homotopies

(iv) Let $f : X \to Y$ be in $\mathcal{C}$. $\gamma(f)$ is an iso iff $f$ is a weak equivalence.

Proof of (iv) $\gamma(f)$ iso $\Rightarrow QR(f)$ is weak equiv

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Dong:

Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $I$ and generating trivial cofibrations $J$.

Let $J$ be a small category. Then $\mathcal{M}^J$ is the category of functors $J \longrightarrow \mathcal{M}$, i.e. $J$-shaped diagrams in $\mathcal{M}$.

There are 2 ways to make $\mathcal{M}^J$ a model category:

Def. A map $F: X \to Y$ in $\mathcal{M}^J$ is a <u>strict widget</u> if for each object $j$ in $J$, the map $F_j : X_j \longrightarrow Y_j$ in $\mathcal{M}$ is a widget.

<u>Def</u> In the <u>projective model category structure on $\mathcal{M}^J$</u>, weak equivalences and fibrations are strict weak equivalences and strict fibrations. A map in $\mathcal{M}^J$ is a cofibration if it has the left lifting property with respect to all trivial fibrations.

In the injective MC on $\mathcal{M}^J$,

cofibrations and weak equivalences
are strict and fibrations

are defined in terms of lifting properties.

<u>Thm</u> These are both MC-structures on $\mathcal{M}^J$ and they are Quillen equivalent.

Recall that the category of $G$-spectra $Sp_G$ is the category of functors from the small category $\mathcal{J}_G$ to $\mathcal{T}_G$, the category of pointed $G$-spaces. i.e. $Sp_G = \mathcal{T}_G^{\mathcal{J}_G}$. We have a CGMC structure in $\mathcal{T}^G$ defined as follows

i) Weak equivalences are equivariant map $X \longrightarrow Y$ that induce weak equivalences (of ordinary pointed spaces) $X^H \longrightarrow Y^H$ for each $H \leq G$. There is a theorem of Bredon that says a map

$f: X \to Y$ of $G$-CW complexes

is an equivariant homotopy equivalence iff $f^H$ is a weak equiv for each $H \leq G$.

ii) The set $I$ of generating cofibrations is $\left\{ G_+ \wedge_H (S^{n-1}_+ \to D^n_+) : \begin{array}{c} n \geq 0 \\ H \leq G \end{array} \right\}$

iii) The set $J$ of generated trivial cofibrations is
$$\left\{ G_+ \wedge_H ( I^n_+ \to I^{n+1}_+ ) : \begin{array}{c} n \geq 0 \\ H \leq G \end{array} \right\}$$

Hence we get a projective MC structure on $Sp^G$, the category of G-spectra an equivariant maps. This is <u>NOT</u> the one we want. We want our weak equivalences to induce isos of <u>stable</u> homotopy groups, not of $\pi_* X_V$ for each $V$.