MZ, CZ, UY, YZ, YQ, QS

M. Zeng on model categories

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Some set theory

Def A set α is an <u>ordinal</u> if

1) Every element is a subset of α
2) α is strictly well ordered with respect to inclusion

e.g. $\emptyset \to \{\emptyset\} \to \{\emptyset, \{\emptyset\}\} \to \{\emptyset, \{\emptyset\}, \underline{2}\} \to \cdots \{\emptyset, \underline{1}, \underline{2}, \cdots \underline{n-1}\}$

$\{\emptyset\} =: \underline{1}$, $\{\emptyset, \{\emptyset\}\} =: \underline{2}$, $\{\emptyset, \{\emptyset\}, \underline{2}\} = \underline{3}$, $\{\emptyset, \underline{1}, \underline{2}, \cdots \underline{n-1}\} = \underline{n}$

$\omega := \operatorname{colim} \underline{n}$

Given an ordinal α, defined $\alpha + 1 = \alpha \cup \{\alpha\}$

Def An ordinal α is a <u>limit ordinal</u> if $\alpha \neq \beta + 1$ for any ordinal β. e.g. $\emptyset$ and ω

Def The <u>cardinality</u> $|A|$ of a set A is the smallest ordinal which bijects to A.

Def An ordinal κ is a <u>cardinal</u> if $|\kappa| = \kappa$

Def Let γ be a cardinal. An ordinal κ is <u>γ-filtered</u> if κ is a limit ordinal and if $A \in \kappa$ and $|A| \leq \gamma$ then $\sup A < \kappa$

Assume all categories in sight are cocomplete

Def Let $\underline{C}$ be a cat and λ an ordinal. A λ-sequence is a colimit preserving function $X : \lambda \to \underline{C}$. Then the transfinite composition of $\underline{X}$ is the map $X_{(0)} \to \operatorname{colim} \underline{X}$.

Def Let C be a category, D a class of maps of C, $A \in \operatorname{ob} C$, and κ a cardinal. Then A is κ-small relative to D if for all κ-filtered ordinals λ and all λ-sequences $\underline{X}$ s.t. $X(\beta) \to X(\beta+1)$ is in D for all β, then $\operatorname{colim}_\beta X(\beta) \to C(A, \operatorname{colim} X(\beta))$ is an isomorphism.

Def Let I be a class of maps in C

1) A map is I-injective if it has RLP w.r.t. to each map in I
2) I-projective ... LLP
3) A map is an I-cofibration if ... LLP for all maps in I $= (I\text{-inj})\text{-proj}$
4) A map is an I-fibration if it RLP ... $= (I\text{-proj})\text{-inj}$

Def Let I be a class of maps in C. A relative I-cell complex is a transfinite composition of pushouts of maps in $\mathcal{A}$, i.e.

i.e. $f: A \to B$ is I-cell if $\exists$ ordinal λ with $X: \lambda \to \mathcal{C}$ with $X(0) = A$, $\forall \beta \in \lambda$ $\exists$ pushout

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta \in I"'] & X(\beta) \arrow[d] \\ D_\beta \arrow[r] & X(\beta+1) \end{tikzcd}$$

and f is the composite of X

for objects $C_\beta, D_\beta \in ob\, \mathcal{C}$

Thm (Small object argument)[SOA] $\mathcal{C}$ a category, I a set of maps in $\mathcal{C}$. Suppose every domain in I is small relative to I-cells. Then there is a functorial factorization (γ, δ) for all maps in $\mathcal{C}$ s.t. $\gamma(f) \in I$-cell and $\delta(f)$ is I-injective.

This thm gives the factorization needed for a MC structure

Def Let $\mathcal{C}$ be a model category. We say it is a cofibrantly generated MC if there are sets of map I and J s.t.

1) domains of maps in I relative to I-cells
2) " J " J-cells
3) Fibrations are J-injective

4) Trivial fibrations are $\downarrow$-injective

and I is the set of generating cofibrations
J trivial cofibrations

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A map is a (trivial) fibration if it has RLP for (I) J.

<u>Prop</u> Let $\mathcal{C}$ be a CGMC with generating sets I and J. Then

1. {cofibs} = I-cofibrations
2. every cofibration is a retract of a relative I-cell complex
3. domains of I are small relative to cofibs

4-6. similarly for trivial cofibs and J.

Proof is long, tedious and omitted.

<u>Thm</u> Let $\mathcal{C}$ be a bicomplete category with a subcategory W (weak equivs) with sets of maps I and J. Then there is a CGMC structure on $\mathcal{C}$ with generating sets I and J iff the following 5 conditions are met.

1) W has 2 out of 3 property and is closed under retracts.

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2) Domains of I are small relative to I-cell
3) " J " J-cell
4) J-cell $\subseteq W \cap I$-cell, i.e. J-cells are triv cofibs
5) I-inj $\subseteq W \cap J$-inj, i.e. I-injectives are trivial fibrations
6) Either $W \cap I$-cof $\subseteq J$-cof or
$W \cap J$-inj $\subseteq I$-injectives.

Proof omitted.

In this case $\{\text{fibrations}\} := J\text{-inj}$
$\{\text{cof}\} := I\text{-cof}$

We use the SOA for factorizations of $f \in \mathrm{Map}(\mathcal{C})$
$f = \beta(f)\alpha(f)$ where $\alpha(f) \in I\text{-cell} \subseteq I\text{-cofib}$
$\beta(f) \in I\text{-inj} \subset$ triv fibns.

Example $\mathcal{C} = \mathcal{T}op$

$$W = \{f : X \to Y \mid \forall x \in X,\ \pi_*(X,*) \longrightarrow \pi_*(Y, f(*)) \text{ is iso}\}$$

$$I = \{\partial D^n \to D^n : n \geq 0$$

$$J = \{D^n \times (\{0\} \hookrightarrow I) : n \geq 0\}$$

Every object is fibrant
I-cells are "relative CW-complexes"

Homotopy category next time.