

CM, YQ, MZ, YZ, UY ITA

# QZ on model categories

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## Basic defs + examples

In homotopy theory, if  $X \simeq Y$ , then  $\forall x X = \pi_x Y$  etc.

Def (Retract of morphism)  $f: X \rightarrow X'$  is a retract of  $g: Y \rightarrow Y'$  if

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{m} & X \\ b \downarrow & & \downarrow g & & \downarrow b \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{m'} & X' \end{array}$$

if  $m_i = 1_X$  and  $m'_i = 1_{X'}$

Def A model category  $\mathcal{C}$  has 3 classes of

morphisms:

weak equivalences  $\mathcal{W}$

fibrations  $\mathcal{Fib}$

cofibrations  $\mathcal{Cof}$

satisfying 5 axioms

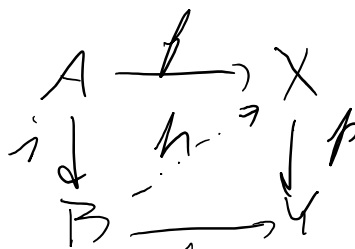
MC1  $\mathcal{C}$  has all <sup>small</sup> limits + colimits, and hence  $\emptyset$  and  $*$ .

MC2 For  $X \xrightarrow{f} Y \xrightarrow{g} Z$ , if 2 of  $f, g$  and  $gf$  are weak eqs, so is the third.

MC3 If  $f$  is a retract of  $g$  and  $g$  is in  $\mathcal{W}, \mathcal{Fib}, \mathcal{Cof}$ ,

then so is  $f$ .

MC4 (Lifting) Given



$\begin{matrix} \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{matrix}$

$\exists h \text{ if } i \in \text{Cof}$  and  $p \in W \cap \text{Fib}$  or  $g \in \text{Cof} \cap W$  and  $p \in \text{Fib}$

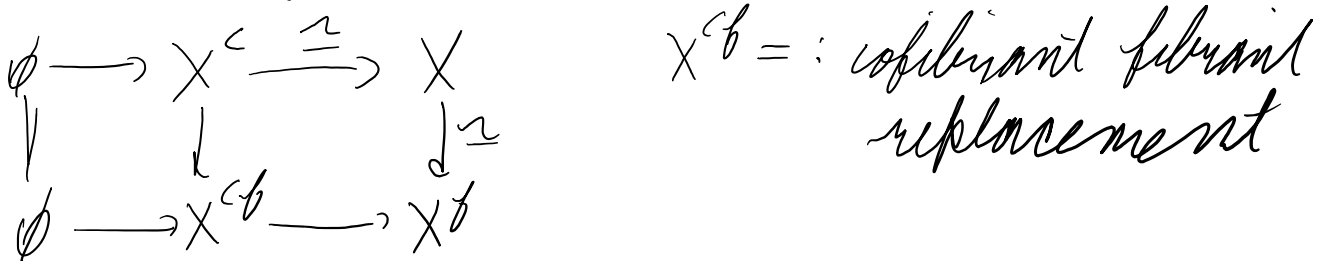
MC 5 Factorization Any morphism  $f$  can be factored as  $f = p \circ i$  where  
 $p = \text{triv fib}$ ,  $i = \text{cofib}$  or  
 $p = \text{fib}$  and  $i = \text{trivial cofib}$ .

Do these imply that isomorphisms are weak equivs?  
 Dwyer-Spalinski assume that identity maps are in  $W \cap \text{Fib} \cap \text{Cofib}$  SEE PAGE →

Def  $X$  is cofibrant if  $\emptyset \rightarrow X$  is cofibration  
 $Y$  is fibrant if  $Y \rightarrow *$  is fibration.

$X^c \rightarrow X$  cofibrant replacement

$Y \rightarrow Y^b$  fibrant replacement



Examples

①  $C = \mathcal{T}_{op \times}$

$W = \{ X \xrightarrow{f} Y : \pi_x(f) \text{ is iso} \}$

$\text{Fib} = \{ X \xrightarrow{f} Y : \forall CW \subset X \subset A, \dots \}$



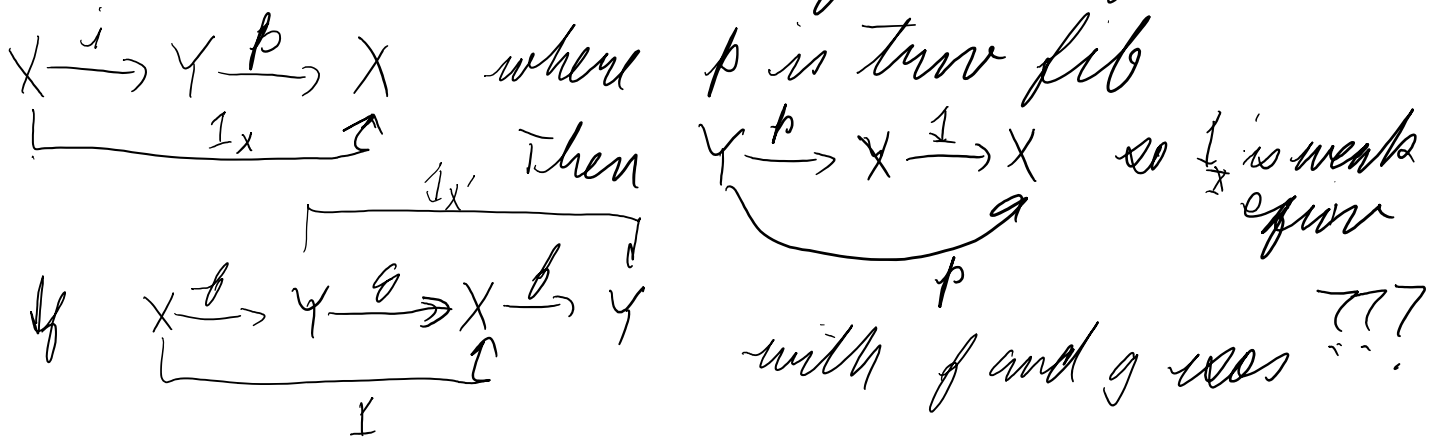
Cof = maps with lifting properties

Verifying MC5 requires small object argument.

Top  $\xrightarrow{\text{sing}}$  sSet. sSet is equiv to a full subcat of  $\mathcal{S}$ , the category of CW-complexes

- ② We use this to define MC structure on sSet.
- Fib =  $\{ X \xrightarrow{f} Y : f \text{ has RLP with respect to horn maps } \Lambda_i^n \rightarrow \Delta^n \}$
- Cof =  $\{ X \rightarrow Y : f: X[n] \rightarrow Y[n] \text{ is 1-1} \}$

To show cof are weak equivs, factor  $\downarrow_X$



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Let  $\mathcal{S}^G$  be category of pointed  $G$ -spaces + equiv maps.

Naive model structure:  $X \xrightarrow{f} Y \in W$  (or Fib) if it is so in  $\mathcal{S}$ . Define Cof by lifting property in  $\mathcal{S}^G$ .

# The genuine MC structure

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$f: X \rightarrow Y \in \mathcal{W}$  on Fib if  $f^H: X^H \rightarrow Y^H$  is  $\forall H \in G$ .

**DIGRESSION** For maps  $X \xrightarrow{f} Y$

$$\begin{array}{ccccc} X & \xrightarrow{1} & X & \longrightarrow & X \\ \downarrow f & & \downarrow 1 & & \downarrow f \\ Y & \xrightarrow{g} & X & \xrightarrow{f} & Y \end{array}$$

as  $f$  is retraction of  $\mathbb{1}_X$   
and hence an equiv.

Cof defined by lifting property

Example: The suspension weak equiv may not be one.

$$X = \{0, 1/n : n \in \mathbb{N}_+\}$$

$$W \xrightarrow{1} X \quad f(n) = \begin{cases} 0 & n=0 \\ 1/n & n>0 \end{cases}$$

This is weak equiv but  $\Sigma f$  is not.

because  $\pi_1 \Sigma W$  is countable

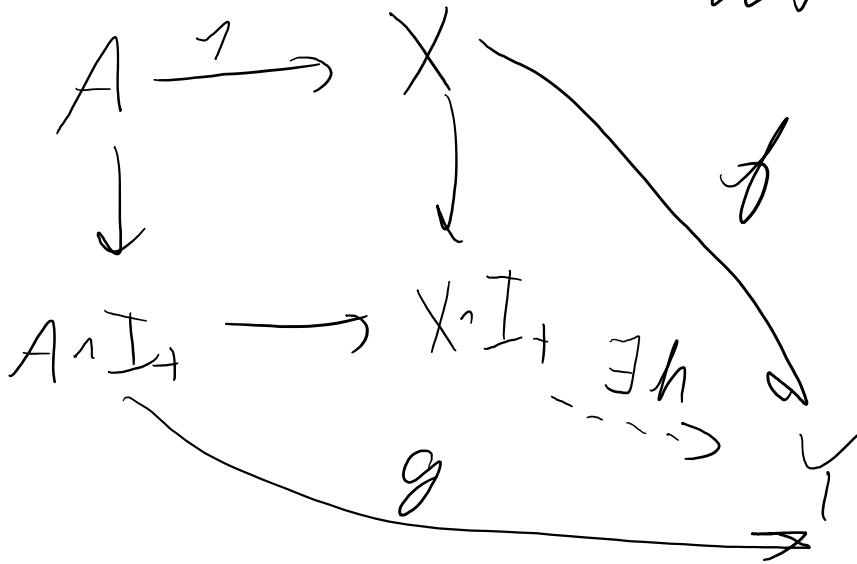
$\pi_1 \Sigma X$  is uncountable

Def Let  $j: S^0 \rightarrow \mathbb{I}_1$  sending nonbase pt to 0

A map  $i: A \rightarrow X$  is  $h$ -cofibration if it is a closed pointed embedding and  $(X, A)$  has Homotopy Extension property, i.e.

For each  $b, g$  with outer diagram commutes,  $\exists h$ .

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Remark

- $\left\{ \begin{array}{l} W = \text{hty equivs} \\ \text{Cof} = h\text{-cofibrations} \end{array} \right.$
- [Fib defined by lifting]

This is Strøm's MC structure on  $\text{Top}_*$ .

Def  $x \in X$  is nondegenerate if  $\{x\} \rightarrow X$  is  $h$ -cofibration in  $\text{Top}_*$ .

Prop If  $X \xrightarrow{f} Y$  is weak equiv with nondegen base points, then  $\Sigma f$  is also a weak equiv.

Why are identity-map fibrations / cofibrations?

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$$\text{Let } \begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{p} & X \\ \downarrow 1 & & \downarrow p & & \downarrow 1 \\ X & \xrightarrow{1} & X & \xrightarrow{1} & X \end{array}$$

is factorization of  $1_X$   
so  $1$  is retract of  $p$

$$\begin{array}{ccccc} X & \xrightarrow{1} & X & \xrightarrow{1} & X \\ \downarrow 1 & & \downarrow i & & \downarrow 1 \\ X & \xrightarrow{i} & Y & \xrightarrow{p} & X \end{array}$$

so  $1$  is retract of  $i$ .