CM, YQ, MZ, YZ, UY TB

QZ on model categories

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Basic defs + examples

In homotopy theory, if $X \simeq Y$ then $\pi_* X = \pi_* Y$ etc.

Def (Retract of morphism) $f: X \to X'$ is a retract of $g: Y \to Y'$ if

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "r'"] & X' \end{tikzcd}$$

if $ri = 1_X$ and $r'i' = 1_{X'}$

Def A model category $\mathcal{C}$ has 3 classes of morphisms:

weak equivalences $\mathcal{W}$
fibrations $\mathcal{F}ib$
cofibrations $\mathcal{C}of$

satisfying 5 axioms

<u>MC1</u> $\mathcal{C}$ has all small limits + colimits, and hence $\emptyset$ and $*$.

<u>MC2</u> For $X \xrightarrow{f} Y \xrightarrow{g} Z$, if 2 of f, g and gf are weak equivs, so is the third.

<u>MC3</u> If f is a retract of g and g is in $\mathcal{W}$, $\mathcal{F}$ or $\mathcal{C}of$, then so is f.

<u>MC4</u> (Lifting) Given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

$B \xrightarrow{g} Y$

$\exists\, h$ if $i \in Cof$ and $p \in W \cap Fib$ or $i \in Cof \cap W$ and $p \in Fib$

MC5 Factorization Any morphism f can be factored as $f = pi$ where
p = triv fib, i = cofib or
p = fib and i = trivial cofib.

Do these imply that isomorphisms are weak equivs?
SEE PAGE →
Dwyer-Spalinski assume that identity maps are in $W \cap Fib \cap Cofib$

Def X is <u>cofibrant</u> if $\emptyset \to X$ is cofibration
Y is <u>fibrant</u> if $Y \to *$ is fibration.
$X^c \to X$ cofibrant replacement
$Y \to Y^f$ fibrant replacement

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X^c \arrow[r, "\simeq"] \arrow[d] & X \arrow[d, "\simeq"] \\
\emptyset \arrow[r] & X^{cf} \arrow[r] & X^f
\end{tikzcd}$$

$X^{cf} =:$ cofibrant fibrant replacement

<u>Examples</u>

① $\mathcal{C} = \mathcal{T}op_*$

$W = \{X \xrightarrow{f} Y : \pi_*(f) \text{ is iso}$

$Fib = \{X \xrightarrow{f} Y : \forall \text{ CW cx } A,$

$$\begin{tikzcd}
A \times \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\
A \times I \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

(Serre fibration)

Cof = maps with lifting property

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Verifying MC5 requires small object argument.

$$\mathrm{Top} \underset{|\cdot|}{\overset{\mathrm{Sing}}{\rightleftarrows}} s\mathrm{Set}$$

. sSet is equiv to a full subcat of $\mathcal{T}$, the category of CW-cxs

(2) We use this to define MC structure on sSet.

Fib $= \{ X \xrightarrow{f} Y :$ f has RLP with respect to horn maps $\Lambda^n_i \to \Delta^n \}$

Cof $= \{ X \to Y ;\ f : X[n] \to Y[n]$ is 1-1 $\}$

DIGRESSION

To show isos are weak equivs, factor 1_X

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "1_X"'] & Y \arrow[r, "p"] & X \end{tikzcd}$$

where p is triv fib

Then

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[rr, bend right, "p"'] & X \arrow[r, "1"] & X \end{tikzcd}$$

so 1_X is weak equiv

If

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "1"'] & Y \arrow[r, "g"] \arrow[rr, bend left, "1_{X'}"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

with f and g isos ???

Let $\mathcal{T}^G$ be category of pointed G-spaces + equiv maps.

<u>Naive model structure</u>: $X \xrightarrow{f} Y \in W$ (or Fib) if it is so in $\mathcal{T}$. Define Cof by lifting property in $\mathcal{T}^G$

The genuine MC structure:

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$f: X \to Y \in W$ or Fib if $f^H: X^H \to Y^H$ is $\forall H \subseteq G$.

DIGRESSION For isos $X \underset{g}{\overset{f}{\rightleftarrows}} Y$

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "1"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

so f is retraction of 1_X and hence an equiv.

Cof defined by lifting property

Example: The suspension weak equiv may not be one.

$X = \{0, 1/n : n \in \mathbb{N}_+\}$

$\mathbb{N} \xrightarrow{f} X \qquad f(n) = \begin{cases} 0 & n = 0 \\ 1/n & n > 0 \end{cases}$

This is weak equiv but Σf is not.

because $\pi_1 \Sigma \mathbb{N}$ is countable

$\pi_1 \Sigma X$ is uncountable

Def Let $j: S^0 \to I_+$ sending nonbase pt to 0

a map $i: A \to X$ is h-cofibration if it is a closed pointed embedding and (X, A) has Homotopy Extension property, i.e.

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For each f, g with outer diagram commutes, $\exists h$.

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$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "f"] & \\
A \wedge I_+ \arrow[r] \arrow[drr, bend right, "g"'] & X \wedge I_+ \arrow[dr, dashed, "\exists h"] & \\
& & Y
\end{tikzcd}$$

Remark

$\begin{cases} W = \text{htpy equivs} \\ Cof = h\text{-cofibrations} \\ Fib \text{ defined by lifting} \end{cases}$

This is Strøm's MC structure on Top_*.

<u>Def</u> $x \in X$ is <u>nondegenerate</u> if $\{x\} \to X$ is h-cofibration in Top_*.

Prop If $X \xrightarrow{f} Y$ is weak equiv with nondegen base points, then Σf is also a weak equiv.

Why are identity map fibrations/cofibs?

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Let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "1"'] & Y \arrow[r, "p"] \arrow[d, "p"] & X \arrow[d, "1"] \\ X \arrow[r, "1"] & X \arrow[r, "1"] & X \end{tikzcd}$$

be factorizn of 1_X

so 1 is retract of p

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[r, "1"] \arrow[d, "i"] & X \arrow[d] \\ X \arrow[r, "i"] & Y \arrow[r, "p"] & X \end{tikzcd}$$

so 1 is retract of i.