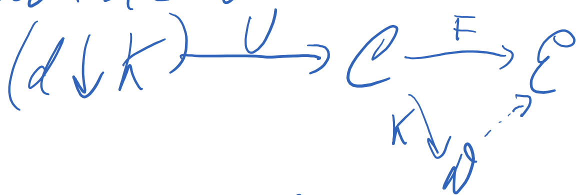
CM, QZ, MZ, SD, QS, YZ

More from Qu

Monday, February 1, 2016 8:59 AM

Thm The right (left) Kan extension of F along K is a pointwise Kan extension iff its value on each $d \in \mathcal{D}$ is the limit (colimit) of $F \cdot U$



where

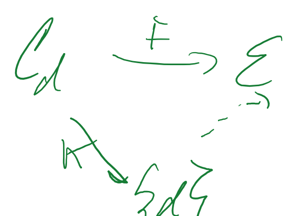
$ob(d \downarrow K) = \{(c, f) : c \in \mathcal{C}, f : d \to Kc \text{ in } \mathcal{D}\}$

Proof follows from definitions.

Ex 1. If $\mathcal{E} = \mathcal{C}$ and $F = 1_\mathcal{C}$, then $Ran_K 1_\mathcal{C}$ is the right adjoint of K

2. If $\mathcal{C}$ is small and $\mathcal{D}$ is trivial (terminal) category then $Ran_\Delta F = \lim F$ and $Lan_\Delta F = \operatorname{colim} F$.

3. Let $\mathcal{C}_d \subset \mathcal{C}$ be $K^{-1}(d)$, then for



the Kan extensions are as in 2.

4. $\mathcal{E} = Vect_k$ for a field k, which is bicomplete

$\mathcal{D} = G$ = gp = cat with one object and invertible morphisms

$\mathcal{C} = H \subseteq G$ "

Then $Lan_K F$ is induced rep

$Ran_K F$ is coinduced rep

A formula for Kan extension

Monday, February 1, 2016 9:14 AM

For $\mathcal{C}$ small and $\mathcal{D}$ (co)complete,

$$\mathrm{Lan}_K F(d) = \int^{c} \mathcal{D}(K(c), d) \cdot F(d)$$ ← copower or tensor

$$\mathrm{Ran}_K F(d) = \int_{c} F(c)^{\mathcal{D}(d, K(c))}$$ ← power

Applying these to ex. 2+3 above gives

$$\text{left adjoint} =: L(d) = \int^{c} \mathcal{D}(K(c), d) \cdot c$$

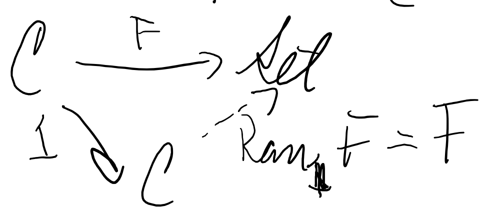
$$\mathrm{colim}\, F = \int^{c} F(c)$$

$$\text{right adjoint} =: R(d) = \int_{c} \mathcal{D}(d, K(c)) \cdot c$$

$$\lim F = \int_{c} F(c)$$

Yoneda reduction + coreduction

Set $\mathcal{C} = \mathcal{D}$, $K = 1_{\mathcal{C}}$, $\mathcal{E} = \mathcal{S}et$



$$F(d) = \int_{c} \mathcal{S}et(\mathcal{C}(d, c), F(c))$$

$$= \int_{c} \mathcal{S}et(h^d(c), F(c))$$

$$= \mathrm{Nat}(h^d(-), F(-))$$

This is the Yoneda lemma

Dually, $\mathrm{Lan}_K F = F$ and $F(d) = \int^{c} \mathcal{C}(c, d) \times F(c)$

Yoneda co-reduction

Def A **monoidal category** $(\mathcal{C}, \square, 1)$ where

Monday, February 1, 2016 9:22 AM

$\mathcal{C} \times \mathcal{C} \xrightarrow{\square} \mathcal{C}$ and $1 \in \mathcal{C}$ with 3 natural transformations

$$(X \square Y) \square Z \cong X \square (Y \square Z)$$

$$1 \square X \cong X \quad \text{and} \quad X \square 1 \cong X$$

with associativity pentagon + unit triangle diagrams commuting

Def A (co)complete monoid category is **(co)Cartesian** if $\square$ is categorical (co)product and 1 is terminal (initial) object

Def A monoidal category $(\mathcal{C}, \square, 1)$ is **symmetric** if $\exists\, \tau_{X,Y} : X \square Y \cong Y \square X$

$$X \square Y = X \square Y, \quad X \square Y \xrightarrow{\tau_{X,Y}} Y \square X \xrightarrow{\tau_{Y,X}} X \square Y$$

Examples $(Set, \times, *)$, $(Top, \times, *)$

Remark When $\mathcal{C}$ is (co)complete, $\square$ need not preserve (co)limits

Ex $(Ab, \otimes, \mathbb{Z})$ does preserve limits

Prop Let $(\mathcal{D}, \oplus, 0)$ be small monoidal cat

$\forall X, Y \in Ob(\mathcal{D})$. $\int^{\mathcal{D}} \mathcal{D}(W \oplus X, Y) \times \mathcal{D}(0, W) \cong \mathcal{D}(X, Y)$

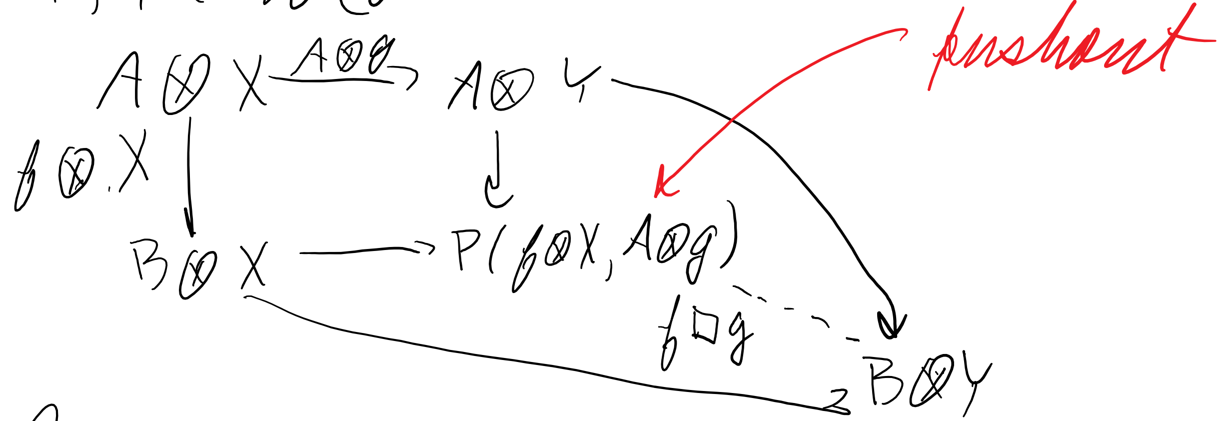
$\forall x, y \in Ob(\mathcal{D})$, $\int^{\mathcal{D}} \mathcal{D}(w \otimes x, y) \times \mathcal{D}(O, w) \cong \mathcal{D}(x, y)$

Def Pushout + pullback corner maps

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be cats with a functor
$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$. (Monoidal if $\mathcal{C} = \mathcal{D} = \mathcal{E}$)

$A, B \in ob(\mathcal{C})$ $\quad A \xrightarrow{f} B$

$X, Y \in ob(\mathcal{D})$ $\quad X \xrightarrow{g} Y$



Dually we defined pullback corner map

$f \lozenge g : A \otimes X \to$ pullback $R(B \otimes g, Y \otimes f)$

Both can be generalized to n-fold operations

Def A monoidal category $(\mathcal{C}, \otimes, 1)$ is __closed__ if $\forall A \in ob\, \mathcal{C}$, the functor $(-)\otimes A$ has a right adjoint $(-)^A$ or $Hom_{\mathcal{C}}(A,-)$ or $\mathcal{C}(A,-)$.
with $\mathcal{C}(B, Hom_{\mathcal{C}}(A,X)) \cong \mathcal{C}(B \otimes A, X)$

$Hom_{\mathcal{C}}$ is the __internal hom__

Consider the arrow cat arr($\mathcal{C}$)

Let $I = \{0 \to 1\}$ so $F : I \to \mathcal{C}$ is a morphism in so $\mathrm{arr}(\mathcal{C}) = \mathcal{C}^I$

If $(\mathcal{C}, \wedge, S)$ is closed monoidal, $\forall a, b \in \mathrm{ob}\,\mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(a,b) \in \mathrm{ob}\,\mathcal{C}$. We define $\otimes$ on $\mathcal{C}^I = \mathrm{arr}(\mathcal{C})$ by $f \otimes g : X_0 \otimes Y_0 \to X_1 \wedge Y_1$ for $f : X_0 \to X_1$, $g : Y_0 \to Y_1$

We show $(\mathrm{arr}\,\mathcal{C}, \otimes, 1_S)$ is closed

$$(\mathrm{arr}\,\mathcal{C})_{\otimes}(f,g) = \underline{\mathcal{C}}(X_0,Y_0) \times_{\underline{\mathcal{C}}(X_0,Y_1)} \underline{\mathcal{C}}(X_1,Y_1) \to \underline{\mathcal{C}}(X_1,Y_1)$$

$$\begin{array}{ccc} \text{pullback} & \longrightarrow & \mathcal{C}(X_1, Y_1) \ni b \\ \downarrow & & \downarrow b^* \\ a \in \mathcal{C}(X_0, Y_0) & \xrightarrow{g_*} & \mathcal{C}(X_0, Y_1) \end{array}$$

$$(1)\quad \begin{array}{ccc} X_0 & \xrightarrow{a} & Y_0 \\ f\downarrow & & \downarrow g \\ X_1 & \xrightarrow{b} & Y_1 \end{array} \quad \leadsto b$$

internal hom in arr($\mathcal{C}$)

arr$\mathcal{C}(f,g)$ is set of diagrams (1)