

# Das on Categories III

Thursday, January 28, 2016 8:30 AM

[skipping filtered and sifted colimits]  
 Let  $J$  be a small cat and  $H: J^{op} \times J \rightarrow \mathcal{C}$

The end  $\int_J H(x, x)$  is the limit (equalizer) of

$$\prod_{x \in \text{ob } J} H(x, x) \xrightarrow[f^*]{b^*} \prod_{f \in \text{arr } J} H(\text{dom } f, \text{cod } f)$$

where  $H(x, x) \xrightarrow{f^*} H(x, y) \xleftarrow{b^*} H(y, y)$

The coend  $\int^J H(x, x)$  is the colimit (coequalizer) of

$$\coprod_{f \in \text{arr } J} H(\text{cod } f, \text{dom } f) \xrightarrow[f^*]{b^*} \coprod_{x \in \text{ob } J} H(x, x)$$

For  $H, H': J^{op} \times J \rightarrow \mathcal{C}$  with  $\theta: H \Rightarrow H'$ , we have

$$\int_J \theta: \int_J H \rightarrow \int_J H' \quad \text{and} \quad (\text{Functoriality of})$$

$$\int^J \theta: \int^J H' \rightarrow \int^J H \quad (\text{ends + coends})$$

There is a Fubini theorem for coends.

Given  $H: J_1^{op} \times J_1 \times J_2^{op} \times J_2 \rightarrow \mathcal{C}$

$$\int_{a \in J_1} \int_{b \in J_2} H(a, a, b, b) = \int_{b \in J_2} \int_{a \in J_1} H(a, a, b, b)$$

$$= \int_{J_1 \times J_2} H(a, a, b, b)$$

and similarly for ends.

Prop 2.4.5 Given  $F, G : J \rightarrow \mathcal{C}$

let  $H(x, y) = \mathcal{C}(F(x), F(y))$  be functors  $J^{\text{op}} \times J \rightarrow \text{Set}$

$$\int_J H(x, y) = \int_J \mathcal{C}(F(x), G(x)) = [J, \mathcal{C}](F, G)$$

= set of nat transformations  $F \Rightarrow G$

### Yonin Qu on More category theory

Let  $\mathcal{C}$  above be Set, so

$$\int_J \text{Set}(J(a, -), G(-)) = \text{Nat}(h^A, G) = G(A) \text{ by Yoneda lemma}$$

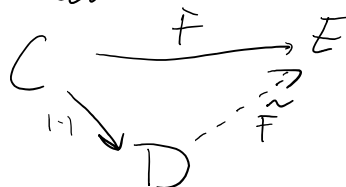
This is the Yoneda reduction

Dually,

$$\int^J J(-, A) \times G(-) \cong G(A) \quad \text{Yoneda co-reduction}$$

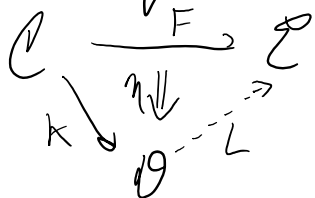
### Kan extensions

Given sets



$\exists \tilde{F}$  but it is not unique

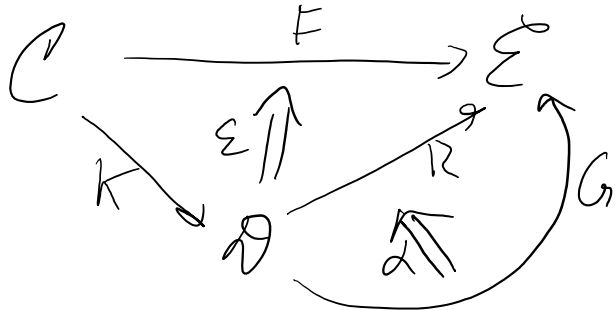
Given categories



If  $\exists$  functor  $L : \mathcal{D} \rightarrow \mathcal{E}$  with nat trans  $\eta : F \Rightarrow KL$  s.t.  $\forall G : \mathcal{D} \rightarrow \mathcal{E}$  with  $\gamma : F \Rightarrow KG$ , then  $\exists ! \alpha : L \Rightarrow G$  such that  $\alpha \eta = \gamma$ , then  $L$  is the left Kan extension of  $F$  along  $K$ .

$$L =: \text{Lan}_K F$$

The definition of right Kan extension is similar but the nat trans goes the other way



We want  $(R, \epsilon)$  s.t  
for any  $(G, \gamma) \exists! \alpha$   
with  $\gamma = \epsilon \alpha$ .

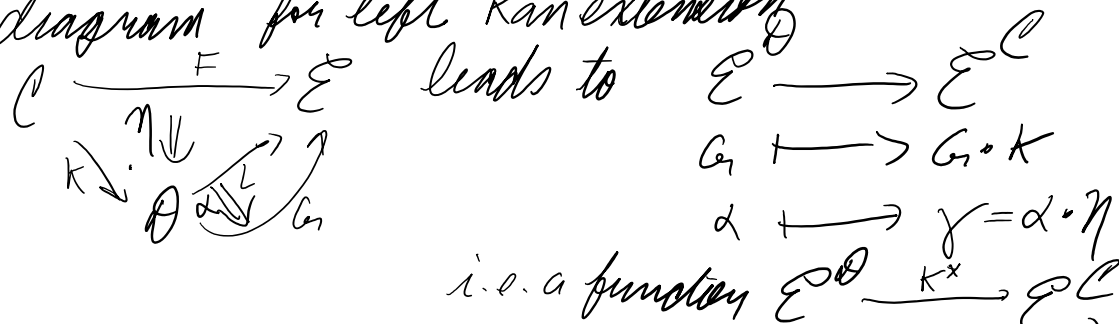
$$\text{Ran}_K F := R$$

This is the right Kan extension.

In a 2-category the objects are functors  $F: C \rightarrow D$   
and morphisms are  $\theta: F \Rightarrow G$

$C$  and  $D$  are fixed, e.g.  $D^C$ , the category  
of all functors  $C \rightarrow D$ . (this defined in Mac Lane)

Our diagram for left Kan extension



i.e. a functor  $E^D \xrightarrow{K^*} E^C$

$$\alpha: \text{Lan}_K F \Rightarrow G \quad \text{so } \alpha \in E^D(\text{Lan}_K F, G)$$

$$\gamma: F \Rightarrow G \circ K \quad \gamma \in E^C(F, G \circ K)$$

Since  $\alpha$  and  $\gamma$  determine each other, we have

$$E^D(\text{Lan}_K F, G) \cong E^C(F, G \circ K) \quad \textcircled{1}$$

This looks like an adjunction.  $\exists \text{Lan}_K F$

for all  $F$ , then  $\text{Lan}_K$  is a functor

In this case  $\mathcal{D}$  is an adjunction

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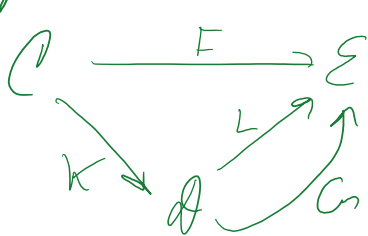
$$\text{Lan}_K \dashv (-) \circ K$$

Similarly

$$(-) \circ K \dashv \text{Ran}_K$$

This is the analogy of the left + right adjoints of  $\Delta$  being  $\text{lim} + \text{colim}$ .

Example  $\mathcal{E} = a \rightarrow b$  which is  $\omega$ -complete

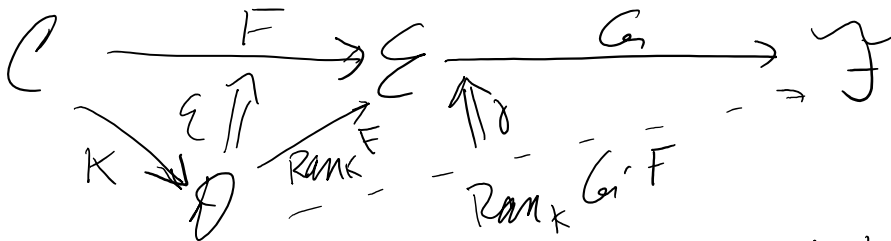


$$\text{Lan}_K(F)(d) = a \quad \forall d$$

but  $G(d)$  could be  $a$  or  $b$

If  $\mathcal{E} = a \rightrightarrows b$  we cannot find a unique natural transformation  $\alpha$  so there is not  $\text{Lan}_K$   
*not complete*

We will discuss things in terms of right Kan extension



Def  $G$  is preserved by right Kan extension if

$$\text{Ran}_K(G \circ F) = G \circ \text{Ran}_K F \quad \text{with } \gamma = G \circ \epsilon$$

Def A functor  $F: \mathcal{E} \rightarrow \text{Set}$  is representable if  
 $\exists$  nat iso  $F \cong h^e := \mathcal{E}(e, -)$

Def A pointwise <sup>(left)</sup> right Kan extension is one that  
 preserves all rep functors  $\mathcal{E} \rightarrow \text{Set}$  ( $\mathcal{E}^{op} \rightarrow \text{Set}$ )

Thm A functor  $F: \mathcal{C} \rightarrow \mathcal{E}$  has a pointwise  
 right Kan extension along  $K: \mathcal{C} \rightarrow \mathcal{D}$  iff  
 $\text{lim}(d \downarrow \mathcal{C}) \xrightarrow{v} \mathcal{C} \xrightarrow{F} \mathcal{E}$  exists  $\forall d \in \mathcal{D}$

$$\begin{array}{ccc}
 (d \downarrow \mathcal{C}) & \rightarrow & \mathcal{C} \xrightarrow{F} \mathcal{E} \\
 & & \searrow \text{---} \\
 & & \mathcal{D}
 \end{array}$$

$K \downarrow$

to be restated Monday .ghj