Das on Categories III

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[skipping filtered and sifted colimits]

Let J be a small cat and $H : J^{op} \times J \to \mathcal{C}$

The *end* $\int_J H(x,x)$ is the limit (equalizer) of

$$\prod_{x \in ob\,J} H(x,x) \underset{f^*}{\overset{f_*}{\rightrightarrows}} \prod_{f \in arr\,J} H(dom\,f, cod\,f)$$

where $H(x,x) \xrightarrow{f_*} H(x,y) \xleftarrow{f^*} H(y,y)$

The coend $\int^J H(x,x)$ is the colimit (coequalizer) of

$$\coprod_{f \in arr\,J} H(cod\,f, dom\,f) \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{x \in ob\,J} H(x,x)$$

For $H, H' : J^{op} \times J \to \mathcal{C}$ with $\theta : H \Rightarrow H'$, we have

$\int_J \theta : \int_J H \to \int_J H'$ and (Functoriality of ends + coends)

$\int^J \theta : \int^J H' \to \int^J H$.

There is a Fubini theorem for coends.

Given $H : J_1^{op} \times J_1 \times J_2^{op} \times J_2 \to \mathcal{C}$

$$\int^{a \in J_1} \int^{b \in J_2} H(a,a,b,b) = \int^{b \in J_2} \int^{a \in J_1} H(a,a,b,b)$$

$$= \int^{J_1 \times J_2} H(a,a,b,b)$$

and similarly for ends.

Prop 2.4.5 Given $F, G : J \longrightarrow \mathcal{C}$

let $H(x,y) = \mathcal{C}(F(x), F(y))$ be functor $J^{op} \times J \to Set$

$$\int_J H(x,x) = \int_J \mathcal{E}(F(x), G(x)) = [J, \mathcal{C}](F, G)$$

= set of nat transformation $F \Rightarrow G$

Yexin Qu on More category theory

Let $\mathcal{C}$ above be Set, so

$$\int_J Set(J(a,-), G(-)) = Nat(h^A, G) =$$

$= G(A)$ by Yoneda lemma

This is the Yoneda reduction

Dually,

$$\int^J J(-, A) \times G(-) \cong G(A)$$

Yoneda coreduction

Kan extensions

Given sets

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "1-1"'] & E \\ D \arrow[ur, dashed, "\tilde{F}"'] & \end{tikzcd}$$

$\exists \tilde{F}$ but it is not unique

Given categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, dashed, "L"'] \arrow[ur, phantom, "\eta\Downarrow", near start] & \end{tikzcd}$$

If $\exists$ functor $L : \mathcal{D} \to \mathcal{E}$ with nat trans $\eta : F \Rightarrow KL$ s.t. $\forall\, G : \mathcal{D} \to \mathcal{E}$ with $\gamma : F \Rightarrow KG$, then $\exists!\, \alpha : L \Rightarrow G$ such that $\alpha\eta = \gamma$, then L is the left Kan extension of F along K.

$L =: Lan_K F$

The definition of right Kan extension is similar but the nat trans goes the other way

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, phantom, "\varepsilon \Uparrow"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "R"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\Uparrow \alpha"'] & \end{tikzcd}$$

We want (R, ε) s.t. for any (G, γ) $\exists! \alpha$ with $\gamma = \varepsilon \alpha$.

$\mathrm{Ran}_K F := R$

This is the right Kan extension.

In a 2-category the objects are functors $F : \mathcal{C} \to \mathcal{D}$ and morphisms are $\theta : F \Rightarrow G$

$\mathcal{C}$ and $\mathcal{D}$ are fixed, e.g. $\mathcal{D}^{\mathcal{C}}$, the category of <u>all</u> functors $\mathcal{C} \to \mathcal{D}$. (This defined in MacLane)

Our diagram for left Kan extension

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & {} \arrow[d, phantom, "\eta \Downarrow"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "L"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\alpha \Downarrow"'] & \end{tikzcd}$$

leads to

$$\mathcal{E}^{\mathcal{D}} \longrightarrow \mathcal{E}^{\mathcal{C}}$$
$$G \longmapsto G \circ K$$
$$\alpha \longmapsto \gamma = \alpha \circ \eta$$

i.e. a functor $\mathcal{E}^{\mathcal{D}} \xrightarrow{K^*} \mathcal{E}^{\mathcal{C}}$

$\alpha : \mathrm{Lan}_K F \Rightarrow G$ so $\alpha \in \mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K F, G)$

$\gamma : F \Rightarrow GK$ $\quad \gamma \in \mathcal{E}^{\mathcal{C}}(F, GK)$

Since α and γ determine each other, we have

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K F, G) \cong \mathcal{E}^{\mathcal{C}}(F, GK) \qquad (1)$$

This looks like an adjunction. If $\exists \mathrm{Lan}_K F$ ~~a~~ for all F, then Lan_K is a functor

In this case $\textcircled{D}$ is an adjunction

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$$\mathrm{Lan}_K \dashv (-)\circ K$$

Similarly

$$(-)\circ K \dashv \mathrm{Ran}_K$$

This is the analog of the left + right adjoints of Δ being lim + colim.

Example $\mathcal{E} = a \to b$ which is cocomplete

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \\ & D \arrow[ur, "L"] \arrow[ur, bend right, "G"'] & \end{tikzcd}$$

$\mathrm{Lan}_K(F)(d) = a \quad \forall d$

but $G(d)$ could be a or b

If $\mathcal{E} = a \rightrightarrows b$ we cannot find a unique natural transformation α.

not complete

so there is not Lan_K

We will discuss things in terms of right Kan extension

$$\begin{tikzcd} C \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\mathcal{E}\Uparrow"] & \mathcal{E} \arrow[rr, "G"] & \arrow[d, phantom, "\Uparrow\gamma"] & \mathcal{F} \\ & D \arrow[ur, "\mathrm{Ran}_K F"] \arrow[urrr, dashed, "\mathrm{Ran}_K G\cdot F"'] & & & \end{tikzcd}$$

Def G is preserved by right Kan extension if

$$\mathrm{Ran}_K(G\cdot F) = G\ \mathrm{Ran}_K F \quad \text{with } \gamma = G\cdot\mathcal{E}$$

Def A functor $F : \mathcal{E} \to Set$ is **representable** if

$\exists$ nat iso $F \cong h^{e} := \mathcal{E}(e, -)$

Def A **pointwise (left) right Kan extension** is one that preserves all rep functors $\mathcal{E} \to Set$ ($\mathcal{E}^{op} \to Set$)

Thm A functor $F : \mathcal{C} \to \mathcal{E}$ has a **pointwise** right Kan extension along $K : \mathcal{C} \to \mathcal{D}$ iff $\lim\, (d \downarrow \mathcal{C}) \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{E}$ exists $\forall d \in \mathcal{D}$

$$\begin{tikzcd}
(d \downarrow K) \arrow[r] & \mathcal{C} \arrow[r, "F"] \arrow[dr, "K"'] & \mathcal{E} \\
 & & \mathcal{D} \arrow[u, dashed]
\end{tikzcd}$$

to be restated Monday .ghj