S. Das on Category Theory II

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Def A **monad** on a category $\mathcal{C}$ is a functor $T: \mathcal{C} \to \mathcal{C}$ with natural transformations $\eta: 1_{\mathcal{C}} \Rightarrow T$ and $\mu: T^2 \Rightarrow T$ such that the following commute

$$
\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[d, Rightarrow, "T\eta"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
$$

Examples 1) $\mathcal{C} = \mathcal{S}et$ $\quad T: A \mapsto \mathcal{P}(A)$ power set

$\eta: a \mapsto \{a\}$

$\mu_A: \mathcal{P}(\mathcal{P}(A)) \to \mathcal{P}(A)$

set of sets $\mapsto$ union of sets

2) Given an adjunction $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ let $T = GF$, η = counit, $\mu = G\varepsilon F$ where ε is the unit of $F \dashv G$.

A **T-algebra** for (T, η, μ) is a pair (X, h) where $X \in \mathcal{C}$ and $h: T(X) \to X$ with

$$
\begin{tikzcd}
T(T(X)) \arrow[r, "T(h)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "h"] \\
T(X) \arrow[r, "h"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "1"'] & T(X) \arrow[d, "h"] \\
 & X
\end{tikzcd}
$$

The T algebras form a category $\mathcal{C}^T$

Example 1) Groups $\mathcal{C} = \mathcal{S}et$ $\quad T(X)$ = underlying set of free gp on X

$\eta_X: X \to T(X)$ $\qquad T(T(X)) \to T(X)$

$x \mapsto [x]$ $\qquad$ homomorphism

A suitable map $h: T(X) \to X$ defines a gp structure on X, so T-algebras are gps.

2) Group actions $\mathcal{C} = \mathrm{Set}$, G a gp

$T(X) = G \times X$ $\qquad \mu : (g_1, g_2, x) \longmapsto (g_1 g_2, x)$

$\eta(x) = (e, x)$

A T-algebra is a G-set.

3) R-modules $\mathcal{C} = \mathcal{A}b$, $T(X) = R \otimes X$

Thm (Eilenberg–Moore construction)
Let (T, η, μ) be a monad on $\mathcal{C}$. Then the forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{C}^T$ sending X to the "free T-algebra" on X.

$X \longrightarrow (T(X), \mu_X)$. The monad for $F \dashv U$ is (T, η, μ). Given another adjunction $F' : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with monad (T, η, μ) $\exists!$ $K : \mathcal{D} \to \mathcal{C}^T$ with $F = KF'$ and $G = UK$.

<u>Limits + colimits</u> Let $X : J \to \mathcal{C}$ be a functor to $\mathcal{C}$ from a small cat J. $j \mapsto X_j$ for $j \in J$ and a morphism $j \xrightarrow{f} j'$ in J gives $X_j \xrightarrow{X_f} X_{j'}$. Then $\mathrm{colim}_J X$ is an object W in $\mathcal{C}$ with maps $w_j : X_j \to W$ with $w_j f = w_{j'}$ and for any other W' with such properties, $\exists!$ $W \to W'$. Limits are dually defined.

Let C^J denote the category of functors $J \to C$
We have constant functor $\Delta : C \to C^J$
then colim : $C^J \to C$ is left adj of Δ.
adjoints of Δ. lim is right adjoint of
$C \longrightarrow C^{J^{op}}$.

Examples 1) J = empty category
$\lim_J = \phi$ = initial object
$\mathrm{colim}_J = *$ = terminal object
2) J = 1 object category for gp G
A functor $J \longrightarrow$ Set or Top defines
a G-action on a set or space X.
$\lim_J X = X^G$ = fixed point space
$\mathrm{colim}_J X = X_G$ = orbit space
3) $J = \bullet \rightrightarrows \bullet$. A functor $J \xrightarrow{F} C$ is
diagram $A \rightrightarrows B$. colim_J =: coequalizer
and $\lim_J$ =: equalizer.
$\lim_J \longrightarrow A \rightrightarrows B \longrightarrow \mathrm{colim}_J$.

Every (co)limit is a (co)equalizer.
Given $X : J \longrightarrow C$ define

$$\prod_{j \in \mathrm{Ob} J} X_j \overset{f}{\underset{g}{\rightrightarrows}} \prod_{\substack{u : j \to k \\ \text{in } J}} X_k$$

by $p_u f = p_k$ and $p_u g = X_u p_j$. The equalizer
is $\lim_J X$. Similarly for colimits.
Left (right) adjoints preserve colimits (limits).

Prop Given adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ and small cat J, we have $F_* : \mathcal{C}^J \rightleftarrows \mathcal{D}^J : G_*$ and the map $\operatorname{colim}_J F_* X \xrightarrow{\cong} F \operatorname{colim}_J$ in $\mathcal{D}$

Proof For $X \in \mathcal{C}^J$ and $Y \in \mathcal{D}^J$

$\mathcal{D}^J(F_* X, Y) = \mathcal{D}^J(FX, Y)$ where $J \xrightarrow{X} \mathcal{C} \xrightarrow{F} \mathcal{D}$

$\mathcal{C}^J(X, G_* Y) = \mathcal{C}^J(X, GY)$ $\quad J \xrightarrow{Y} \mathcal{D} \xrightarrow{G} \mathcal{C}$

Let $\theta' \in \mathcal{D}^J(FX, Y)$, a natural trans $FX \Rightarrow Y$

$\theta \in \mathcal{C}^J(X, GY)$ $\quad$ " $\quad X \Rightarrow GY$

$\theta_j : X_j \to G Y_j \longleftrightarrow \theta_j' : F X_j \longleftrightarrow Y_j$ since $F \dashv G$.

Claim this diagram commutes

$$\begin{tikzcd}
 & \mathcal{D}^J \arrow[dl, shift left, "G_*"] \arrow[dr, shift left, "\operatorname{colim}"] & \\
\mathcal{C}^J \arrow[ur, shift left, "F_*"] \arrow[dr, shift left, "\operatorname{colim}"] & & \mathcal{D} \arrow[ul, shift left, "\Delta"] \arrow[dl, shift left, "G"] \\
 & \mathcal{C} \arrow[ul, shift left, "\Delta"] \arrow[ur, shift left, "F"] &
\end{tikzcd}$$

For $X \in \mathcal{C}^J$ and $Y \in \mathcal{D}$

$\mathcal{C}^J(X, G_* \Delta Y) \cong \mathcal{D}^J(F_* X, \Delta Y)$

$\cong \mathcal{C}(\operatorname{colim}_J F_* X, Y)$

Hence

$F_* \operatorname{colim}_J \dashv \Delta G$

so $F_* \operatorname{colim}_J$ and $\operatorname{colim}_J F_*$ are adjoint to the same functor and are therefore equal QED

A Reflexive coequalizer is the colimit of

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xleftarrow{s} \quad \text{with } fs = gs = 1_B$$

A "reflexive equalizer is its limit.

Such a diagram is a functor from $J = \cdot \rightleftarrows \cdot$

It has subcats End_A and $\tilde{J}$ $\cdot \rightrightarrows \cdot$

Then $\operatorname{colim}_J = \operatorname{colim}_{\tilde{J}} = \operatorname{colim}_{\operatorname{End}_A}$