S. Das on Category theory

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Categories are assumed to be locally small.

<u>Comma Category</u> Given functors $\mathcal{A} \xrightarrow{S} \mathcal{C}$, $\mathcal{B} \xrightarrow{T} \mathcal{C}$. Comma category $S \downarrow T$ has

objects (α, f, β) where $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $f : S(\alpha) \to T(\beta)$

morphisms where $g : \alpha \to \alpha'$, $h : \beta \to \beta'$

$(\alpha, f, \beta) \xrightarrow{(g,h)} (\alpha', f', \beta')$ with

$$\begin{tikzcd} S(\alpha) \arrow[r, "S(g)"] \arrow[d, "f"'] & S(\alpha') \arrow[d, "f'"] \\ T(\beta) \arrow[r, "T(h)"'] & T(\beta') \end{tikzcd}$$

The <u>slice category</u> is special case where

$\mathcal{A} = \mathcal{C}$, $S = 1_{\mathcal{C}}$, $\mathcal{B} = 1$ (trivial category) $T = \gamma \in \mathcal{C}$

$(S \downarrow T) := (\mathcal{C} \downarrow \gamma) =$ category of objects in $\mathcal{C}$ over γ.

objects (α, f) $\alpha \xrightarrow{f} \gamma$ morphisms
$$\begin{tikzcd} \alpha \arrow[rr, "g"] \arrow[dr, "f"'] & & \alpha' \arrow[dl, "f'"] \\ & \gamma & \end{tikzcd}$$

Can define <u>coslice category</u> (objects under γ) similarly

<u>Arrow category</u> $(1_{\mathcal{C}} \downarrow 1_{\mathcal{C}}) = \vec{\mathcal{C}}$ with $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and $S = T = 1_{\mathcal{C}}$

objects are morphisms $\alpha \xrightarrow{f} \beta$ in $\mathcal{C}$

morphisms are commuting squares.

We have functors

$$\mathcal{B} \xleftarrow{\text{codomain}} (S \downarrow T) \xrightarrow{\text{domain}} \mathcal{A}$$

$$\beta \longleftarrow\!\shortmid (\alpha, f, \beta) \longmapsto \alpha$$

$$S \downarrow T \longrightarrow \vec{\mathcal{C}}$$

$$(\alpha, f, \beta) \longmapsto f$$

A <u>natural transformation</u> $\theta : F \Rightarrow G$ for functors $F, G : \mathcal{C} \to \mathcal{D}$ assigns for each object x in $\mathcal{C}$ a morphism $\theta_x \in \mathcal{D}(F(x), G(x))$ such that for each morphism $X \xrightarrow{f} Y$ the diagram

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$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd}$$

commutes

Let $Nat(F,G)$ denote the set of all such.

Yoneda Lemma For an object A in $\mathcal{C}$ let $h^A : \mathcal{C} \to Set$ be $\mathcal{C}(A,-)$

Given another covariant functor $F : \mathcal{C} \to Set$,

$Nat(h^A, F) = F(A)$

Proof: For $f : A \to X$ in $\mathcal{C}$ and $\theta \in Nat(h^A, F)$

$$\begin{tikzcd} 1_A \arrow[d, mapsto] & h^A(A) \arrow[r, "\theta_A"] \arrow[d, "h^A(f)"'] & F(A) \arrow[d, "F(f)"] & \theta_A(1_A) =: k(\theta) \arrow[d, mapsto] \\ f & h^A(X) \arrow[r, "\theta_X"] & F(X) & \theta_X(f) = F(f)(k(\theta)) \end{tikzcd}$$

so θ is determined by $k(\theta)$, where

$$\begin{aligned} k : Nat(h^A, F) &\longrightarrow F(A) \\ \theta &\longmapsto \theta_A(1_A) \end{aligned}$$

QED

Yoneda embedding $Y : \mathcal{C}^{op} \longrightarrow Set^{\mathcal{C}}$

$A \longmapsto h^A$

Yoneda lemma says

$[\mathcal{C}, Set](h^A, F) = F(A)$

Adjoint functors and monads

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Given a functor $F: \mathcal{C} \to \mathcal{D}$, its left/right adjoints are functors $L, R: \mathcal{D} \to \mathcal{C}$ if

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, RY) \text{ and}$$

$$\mathcal{C}(LY, X) \cong \mathcal{D}(Y, FX).$$ We denote this by $F \dashv R$ and $L \dashv F$.

Prop 2.2.5 Given $\mathcal{C} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathcal{E}$ with $F_1 \dashv G_1$ and $F_2 \dashv G_2$. Then $F_2F_1 \dashv G_1G_2$

Proof Let $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $Z \in \mathcal{E}$

$$\mathcal{E}(F_2F_1X, Z) \cong \mathcal{D}(F_1X, G_2Z) \cong \mathcal{C}(X, G_1G_2Z)$$

QED

Prop 2.2.6 Adjoints are unique up to natural equivalence, i.e. $L_1 \dashv F$ and $L_2 \dashv F \Rightarrow L_1 \cong L_2$.

Given $F: \mathcal{C} \rightleftarrows \mathcal{D} : G$ with $F \dashv G$, the counit $\eta: 1_{\mathcal{C}} \Rightarrow GF$ is defined by

$$\mathcal{C}(X, GY) \cong \mathcal{D}(FX, Y)$$

if $Y = FX$ we have $\mathcal{C}(X, GFX) \cong \mathcal{D}(FX, FX)$ so η_X is the map $X \xrightarrow{\eta_X} GFX$ corresponding to 1_{FX}. Similarly the unit $\varepsilon: FG \Rightarrow 1_{\mathcal{D}}$

Thm 2.2.8

The characterization of adjoint ~~functions.~~ functors

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$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is characterized by

counit $\eta : 1_{\mathcal{C}} \Rightarrow GF$

unit $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ with

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[rr, bend right, "id"'] & GFG \arrow[r, Rightarrow, "G\varepsilon"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[rr, bend right, "id"'] & FGF \arrow[r, Rightarrow, "\varepsilon F"] & F \end{tikzcd}$$

Examples

(1) $Ab \xrightarrow{G} Set$ forgetful functor

$Set \xrightarrow{F} Ab$ free abelian gp functor

For $X \in Set$, $\eta_X : X \to GF(X)$ inclusion of generators

For $Y \in Ab$, $\varepsilon_Y : FG(Y) \to Y$ homomorphism induced by generators

$Ab(FX, Y) \cong Set(X, GY)$ so $F \dashv G$

(2) $F : Top \to Top$ $\quad G : Top \to Top$

$X \mapsto X \times I$ $\quad Y \mapsto Top(I, Y) =: PY$

Then $F \dashv G$

$Top(FX, Y) \cong Top(X, PY)$

$Top(\underset{\parallel}{X \times I}, Y) \cong Top(X, Top(I, Y))$

counit $\eta_X : X \longrightarrow Top(I, I \times X)$

$x \longmapsto (t \mapsto (t, x))$ constant path

unit $\varepsilon_Y : I \times Top(I, Y) \longrightarrow Y$ evaluation map

(3) Set^G = category of G-sets + equivariant maps

$\Delta : Set \longrightarrow Set^G$ trivial action

$Set^G(\Delta X, Y) = Set(X, Y^G)$ so

$\Delta \dashv (-)^G$ fixed point set

Similarly $(-)_G \dashv \Delta$ orbit set

(4) Given a subgp $H \subseteq G$ we have

$i_H^G : Set^G \longrightarrow Set^H$ forgetful functor

and $ind_H^G : Set^H \longrightarrow Set^G$

$X \longmapsto G \times_H X = (G \times X)_H$

where $h(g, x) = (gh, hx)$

Then $Set^G(G \times_H X, Y) = Set^H(X, i_H^G Y)$

(5) Choose a cat $\mathcal{C}$ and let Cat be the category of categories. Define $F, G : Cat \to Cat$ by $F(-) = \mathcal{C}^{op} \times -$

$G(-) = [\mathcal{C}^{op}, -]$ Then $F \dashv G$ since

$$[\mathcal{C}^{op} \times \mathcal{A}, \mathcal{B}] \cong [\mathcal{A}, [\mathcal{C}^{op}, \mathcal{B}]]$$

For $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = Set$ we have

$$[\mathcal{C}^{op} \times \mathcal{C}, Set] = [\mathcal{C}, [\mathcal{C}^{op}, Set]]$$

$\mathcal{C}(-,-) \longleftrightarrow$ Yoneda embedding

Coevaluation functor

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left adjoint of evaluation

Fix a category $\mathcal{C}$ with object A

$$F^A : Set \longrightarrow [\mathcal{C}, Set]$$

$$X \longmapsto h^A(-) \times X$$

Endomorphism Category for $A \in \mathcal{C}$

End_A has one object A and morphisms $\mathcal{C}(A,A)$

It is a monoid

Corestriction functor

$$G^A : [End_A, Set] \longrightarrow [\mathcal{C}, Set]$$

Set X with monoid action $\longmapsto h^A(-) \times_{\mathcal{C}(A,A)} X$

left adjoint of $res_A : [\mathcal{C}, Set] \longrightarrow [End_A, Set]$