Let G be a finite gp. Want to define its orthogonal rep ring $RO(G)$

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Let V and W be finite dimensional reps of G. Then $V \oplus W$ and $V \otimes W$ are also reps of G. We get addition + multiplication on the set of iso classes of such reps. $V \otimes (W_1 \oplus W_2) = (V \otimes W_1) \oplus (V \otimes W_2)$. Hence we get a semi-ring. We can introduce subtraction formally and get a ring. Can do the same over $\mathbb{C}$ and get a ring $R(G)$. See Serre's book: Linear reps of finite gps.

Thm Additively $R(G)$ is a free abelian gp whose rank is the # of conjugacy classes of elements of G, e.g. $|G|$ for abelian G, and 3 for $G = S_3$.

One can define a C_2 action on $R(G)$ related to complex conjugation.

$RO(G) \cong$ orbit space of this action.

Def A rep is *irreducible* if it cannot be decomposed as a direct sum.

The # of such over $\mathbb{C}$ is the # of conj classes

There is a C_2-action on the set of conj classes defined by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in G$.

The # of real irreducible reps is the # of orbits under this action.

e.g. $G = C_4$ with generator γ. Then there are 4 conjugacy classes

$1, \gamma, \gamma^2, \gamma^3$, so C_4 has 3 irreducible real reps.

<u>Def</u> A virtual rep of G is an expression, $V-W$ where V, W are actual reps. $V-W = V'-W'$
$\iff V+W' = V'+W$

A G-spectrum X has $RO(G)$ graded homotopy gps, defined as follows.

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For an actual rep V, $\pi_V(X) = [\Sigma^\infty S^V, X]^G$

= gp of homotopy classes of maps

$$\Sigma^\infty S^V \longrightarrow X$$

For a virtual rep $V-W$,

$$\pi_{V-W}(X) = [\Sigma^\infty S^V, X \wedge S^W]^G$$

(this depends only on $V-W$, not on V or W)

Back to the main theorem. We have a spectrum Ω with certain properties.

To construct it, start with $N_2^8 MU_{\mathbb{R}} = X$

There is a certain element $D \in \pi_{19\rho_8} X$

where ρ_8 = regular rep of C_8. We have a map

$$\underset{\substack{\| \\ S^{-0} \wedge S^{19\rho_8}}}{\Sigma^\infty S^{19\rho_8}} \xrightarrow{f} X.$$

X is a commutative ring spectrum (to be defined later). Smashing with X we get

$$\underset{\substack{\| \\ S^{19\rho_8} \wedge X}}{\Sigma^{19\rho_8} X} \xrightarrow{f \wedge X} X \wedge X \xrightarrow{mult} X$$

For each rep V, there is a spectrum (to be defined later) S^{-V}. There is a map $S^{-V} \wedge S^V \longrightarrow S^{-0}$ which is a weak equivalence.

Let $V = 19\rho_8$ and $X = N_2^8 MU_{\mathbb{R}}$

We have $S^V \wedge X \longrightarrow X$

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One can define a smash product of spectra in a nice way (LATER)

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We have $X \wedge S^V \longrightarrow X$

$$\begin{tikzcd} S^{-V} \wedge X \wedge S^V \arrow[r] \arrow[d, "\simeq"] & S^{-V} \wedge X \\ X & \end{tikzcd}$$

We can iterate and get

$$X \longrightarrow S^{-V} \wedge X \longrightarrow S^{-2V} \wedge X \longrightarrow \cdots\cdots$$

Let $\tilde{\Omega}$ be the direct limit (to be defined later) $\pi_\star \tilde{\Omega} = D^{-1} \pi_\star X$

star

This is a C_8-spectrum

G-spectra have fixed point spectra, to be defined later.

$\Omega = \tilde{\Omega}^{C_8}$ = C_8-fixed point spectrum for $\tilde{\Omega}$.

Ω has 3 properties

1) Detection Theorem. There is a map $S^{-0} \longrightarrow \Omega$ which detects θ_j if it exists

2) Periodicity theorem: $\Sigma^{256} \Omega \simeq \Omega$

3) Gap theorem: $\pi_{-2} \Omega = 0$.

3) is the most interesting + uses new methods. We need the slice SS.

Recall the Postnikov construction

By attaching cells to X we can kill its π_i for $i > n$ and get a space $P^n X$, its nth Postnikov section. The fiber of the map $X \longrightarrow P^n X$ is $P_{n+1} X$, the n-connected cover of X

for $i \leq n$

of the map $X \to P^n X$ or $P_{n+1} X$)

n-connected cover of X

$$\pi_i P^n X = \begin{cases} \pi_i X & \text{for } i \le n \\ 0 & \text{for } i > n \end{cases} \qquad \pi_i P_{n+1} X = \begin{cases} 0 & \text{for } i \le n \\ \pi_i X & \text{for } i > n \end{cases}$$

There are maps
$P^{n+1}X \to P^n X \to P^{n-1}X \to P^{n-2}X \to \cdots\cdots$

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The fiber of $P^n X \to P^{n-1}X$ is denoted by $P_n^n X$. $\pi_i P_n^n X = \begin{cases} \pi_i X & \text{for } i = n \\ 0 & \text{for } i \neq n \end{cases}$

Consider the category of n-connected spaces $\mathcal{T}_{>n}$. It is generated in some sense by S^{n+1}, or by the set $T_{n+1} = \{S^{n+1}, S^{n+2}, S^{n+3}, \ldots\}$ under wedges mapping cones and colimits.

There is formal machinery (due to Dror-FARJOUN 1996) by which these subcategories determine the functors P^n, P_{n+1} and P_n^n.

Let G be a finite gp and $H \subseteq G$ a subgp

Consider

SLICE CELLS $W_{m,H} := G_+ \underset{H}{\wedge} S^{m\rho_H} \simeq \bigvee_{|G/H|} S^{m|H|}$

Let $T_n = \{W_{m,H} : m|H| \geq n\}$

We can use this to define functors P_{n+1}, P^n and P_n^n as before for a G-space X.

In the classical case we have a tower

$$\begin{tikzcd} P^0X & P^1X \arrow[l] & P^2X \arrow[l] & P^3X \arrow[l] & \cdots \arrow[l] \\ & P^1_1X \arrow[u] & P^2_2X \arrow[u] & P^3_3X \arrow[u] & \end{tikzcd}$$

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This leads to a SS with

$$E_1^{s,t} = \pi_* P^s_s X$$

It is useless because its input is $\pi_* X$.

We can do a similar thing with spectra

For $MU_{\mathbb{R}}$ and its relatives one can identify the slices $P^n_n X$ as G spectra related to $H\mathbb{Z}$ which have interesting homotopy gps.