Lectures next week
Monday - Shouman on 2.2-2.4
Wednesday - Yexin on 2.5-2.7
Thursday - Ugur on 2.8-2.11

Some useful categories
$\mathcal{T}op$ = category of compactly generated weak Hausdorff spaces
$\mathcal{T}$ = category pointed spaces as above
G = finite gp
$\mathcal{T}^G$ and $\mathcal{T}_G$ both have pointed G-spaces as objects
In $\mathcal{T}^G$ the morphisms are all equivariant continuous pointed maps.

<u>Notation</u> For a category $\mathcal{C}$ with objects X and Y,
$\mathcal{C}(X,Y)$ = set of morphisms $X \to Y$
$\mathcal{T}^G(X,Y)$ has a natural topology, i.e. it is an object in $\mathcal{T}$
In $\mathcal{T}_G$, <u>all</u> cont. pointed maps are morphisms $\mathcal{T}_G(X,Y)$ is a G-space
i.e. for $X \xrightarrow{f} Y$ and $\gamma \in G$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\gamma"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

← need not commute

We define $\gamma(f) = \gamma^{-1} f \gamma$
Recall for $H \subset G$ and an H-space X, we can form a G-space $G_+ \underset{H}{\wedge} X$. For a G-space Y we have

(1) $\mathcal{T}^G(G_+ \underset{H}{\wedge} X, Y) \cong \mathcal{T}^H(X, i_H^G Y)$

where $i_H^G : \mathcal{T}^G \to \mathcal{T}^H$ is the forgetful functor. Note that the ordinary space underlying $G_+ \underset{H}{\wedge} X$ is $\bigvee_{|G/H|} X$

(1) is an example of an <u>adjoint functor</u>

$$\begin{tikzcd} \mathcal{T}^H \arrow[r, shift right, "G_+ \wedge_H (-)"'] & \mathcal{T}^G \arrow[l, shift right, "i_H^G"'] \end{tikzcd}$$

left adjoint — $G_+ \underset{H}{\wedge}(-)$

i_H^G ← right adjoint

Notation: Given categories + functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

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such that for objects X in $\mathcal{C}$ and Y in $\mathcal{D}$,

$$\mathcal{C}(X, G(Y)) \cong \mathcal{D}(F(X), Y)$$

we say they are adjoint

$F \dashv G$ \dashv

We can do similar things in the (yet to be defined) category of spectra $\mathcal{S}^G$, $\mathcal{S}_G$.

The star of our show

$MU_{\mathbb{R}}$, a C_2-spectrum

MU = complex cobordism spectrum

$U(n)$ = n-dimensional unitary gp

$BU(n)$ = classifying space, which can described in terms of complex Grassmannians

It has a $\mathbb{C}^n$-bundle ξ_n with with a Thom space $MU(n)$.

Def (original) A spectrum E is a a collection of spaces E_n and maps $\varepsilon_n : \Sigma E_n \to E_{n+1}$

A map of spectra $E \xrightarrow{f} F$ is a collection of maps $f_n : E_n \to F_n$ such that

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\varepsilon} & E_{n+1} \\ \downarrow f_n & & \downarrow f_{n+1} \\ \Sigma F_n & \xrightarrow{\varepsilon} & F_{n+1} \end{array}$$

$$\pi_k E = \varinjlim_n \pi_{n+k} E_n$$

f is weak equivalence of spectra if it induces iso $\pi_* E \to \pi_* F$.

The spectrum MU is defined by

$$MU_{2n} = MU(n)$$

$$MU_{2-n+1} = \Sigma MU(-n)$$

The map $U(n) \hookrightarrow U(n+1)$ leads to

$$S^2 \wedge MU(n) = \Sigma^2 MU(n) \longrightarrow MU(n+1).$$

We can define an action of C_2 on the spaces $U(n)$, $BU(n)$ and $MU(n)$ in terms of complex conjugation.

The resulting C_2-spectrum is called $MU_{\mathbb{R}}$.

Recall we had a second functor

$$\mathcal{J}^H \longrightarrow \mathcal{J}^G$$

$$X \longrightarrow \mathrm{Map}^H(G_+, X)$$

We get a space underlain by

$$X^{\wedge(G/H)}$$

On the spectrum level we get the norm functor

$$N_H^G : \mathcal{S}^H \longrightarrow \mathcal{S}^G$$

e.g. $H = C_2$, $G = C_8$ consider

$$N_{C_2}^{C_8}(MU_{\mathbb{R}})$$

Tentative definition of a G-spectrum
We want a collection of G-spaces $\{E_V\}$ where V is a finite dimensional vector space with orthogonal G-action. Suppose $V \subset W$ with orthogonal complement $W-V$.
We have structure maps

$$S^{W-V} \wedge E_V \longrightarrow E_W$$

S^{W-V} = one point compactification of $W-V$.

Back to $MU_{\mathbb{R}}$: $G = C_2$
Let σ be the sign rep of C_2
1 " trivial "
1-dim

$\rho = 1 + \sigma$ = regular rep of C_2

The regular rep ρ_G of a gp G is $\mathbb{R}[G]$ = real gp ring of G.

In $MU_{\mathbb{R}}$, $MU(n)$ is the space indexed by $n\rho \cong \mathbb{C}^n$. The others will be revealed later.

An example of a G-spectrum: the sphere spectrum S^{-0}

Let

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$$(S^{-0})_V = S^V = V \cup \{\infty\}$$

$$S^{W-V} \wedge S^V \longrightarrow S^W$$

given by the iso $W-V \oplus V \cong W$

Given a spectrum E and a space X, we define a spectrum $E \wedge X$ by

$$(E \wedge X)_V = E_V \wedge X.$$

e.g. $E = S^{-0}$ then $S^{-0} \wedge X$ is also known as $\Sigma^\infty X$, the suspension spectrum of the space X.