Outline of the proof of the Kervaire invariant theorem

There are certain hypothetical elements

$\theta_j \in \pi_{2^{j+1}-2}$ S^0 ← sphere spectrum

It is related to the Kervaire-Milnor classification of differentiable structures on spheres. (~1963)

1957: Milnor ∃ 7-manifolds Σ^7 that are homeomorphic but not diffeomorphic to S^7

1960: Kervaire ∃ topological 10-manifold with no differentiable structure.

1963: K + M almost classified diff structures on S^k for $k \geq 5$, in terms of $\pi_{n+k} S^n$ for $n >> 0$, except for an ambiguous factor of 2 for $k \equiv 1$ mod 4. Either a) ∃ an exotic S^{4k+1} or b) ∃ framed $(4k+2)$-mfd not cobordant to S^{4k+2}. This is the Kervaire invariant problem.

1969 Browder: Option a) occurs unless $k = 2^{j-1} - 1$ for some k

In those cases, option b) occurs only if h_j^2 in the Adams spectral sequence survives to E_∞

$|h_j^2| = 4k+2 = 2^{j+1}-2$

If h_j^2 survives we get $\theta_j \in \pi_{2^{j+1}-2}(S^{-0})$.
These are known to exist for $1 \le j \le 5$.
In the 1970s many people tried to construct θ_j for all j and <u>failed</u>.

<u>Thm</u> (Hill-Hopkins-R 2009)
$\nexists\, \theta_j$ for $j \ge 7$.

The case $j = 6$ is still open.
How did we do it?
We construct a ring spectrum Ω with 3 properties

1) <u>Detection Theorem</u>. The unit map $S^{-0} \to \Omega$ is such that if $\exists\, \theta_j \in \pi_{2^{j+1}-2} S^{-0}$, its image in $\pi_{2^{j+1}-2}\Omega$ is nontrivial.
2) <u>Periodicity Theorem</u> $\pi_k \Omega$ depends only on k mod 256.
3) <u>Gap Theorem</u> $\pi_k \Omega = 0$ for $-4 < k < 0$, e.g. $\pi_{-2}\Omega = 0$.

2) and 3) $\Longrightarrow \pi_{254}\Omega = 0$ and $\dim \theta_7 = 254$

How to construct Ω and show it has these 3 properties???

The construction involves equivariant stable homotopy theory.
Suppose we have a group G acting on a space X (e.g. C_8), i.e. for each $\gamma \in G$ we have a homeo $f_\gamma : X \to X$ with $f(\gamma\gamma') = f_\gamma f_{\gamma'}$.
Each subgp $H \subset G$ also acts on X. We have $X^H := \{x \in X : \gamma(x) = x \text{ for all } \gamma \in H\}$

$= $ fixed point space of H.

$X_H := X / x \sim \gamma(x)$ for $\gamma \in H$

$=$ orbit space of H.

Our spectrum Ω is $\tilde{\Omega}^G$ for $G = C_8$

and $\tilde{\Omega}$ a C_8-spectrum to be named later.

<u>Example</u> Let V be a finite dimensional orth real vector space with an orthogonal action of G. For each $H \subseteq G$, V^H is a Euclidean vector space. Suppose $\dim V = d$. Let $S^V =$ one point compactification of V. It is a G-space where G fixes ∞.

$$(S^V)^H = S^{(V^H)}$$

WHAT IS A G-SPECTRUM???

See my talk of May 2015.

<u>Construction</u> Let $H \subseteq G$ and let X be an H-space (a space acted on by H). Will construct a space Y on which G acts in two different ways

1) $Y = G \times_H X = G \times X / (g\gamma, x) \sim (g, \gamma(x))$

for $g \in G$ and $\gamma \in H$.

As a space, $Y = G/H \times X$ finite set

2) $Y' = H\text{-maps}\,(G, X)$ where H acts on X as defined and on G by right

multiplication. This is a subspace of $\mathrm{Map}(G, \underline{X}) \approx X^{|G|}$ homeo. (as an ordinary space) to $X^{|G/H|}$

Both constructions have pointed analogs, i.e. assume all spaces in sight have base points fixed by the gp (H or G)

Then 1) becomes

$$G_+ \wedge_H \underline{X} \quad \text{where } G_+ = G \amalg pt$$

$$\approx \bigvee_{|G/H|} \underline{X}$$

2) We get a subspace of $X^{\wedge|G|}$ homeo to $\underline{X}^{\wedge|G/H|}$

$|G|$-fold smash product