Prop 7.1 ≥ The image of

$$\begin{tikzcd}
\pi_V^G EC_{2+} \wedge P_{|V|-1} R(2^\ell-2) \arrow[dr] \arrow[dd] & \\
& \pi_V^G EC_{2+} \wedge R(2^\ell-2) \arrow[dl] \\
\pi_V^G R(2^\ell-2) \arrow[d] & y_\ell \arrow[u, phantom, sloped, "\in"] \arrow[d] \\
\pi_{|V|}^u R(2^\ell-2) & \pi_0^\star N\bar{M}_{2^\ell-1}
\end{tikzcd}$$

is contained in that of $(1-\gamma)$, but the image of y_ℓ is not. [This implies the Reduction Theorem.]

$V = (2^\ell-1)\rho_G$

ρ_G = reg rep of $G = C_{2^n}$

$\ell \geq 1$

γ = gen of G

Proof

$$\begin{tikzcd}
\pi_V^G \left(EC_{2+} \wedge P_{|V|} R(2^\ell-2)\right) \arrow[d] \arrow[drr, dashed] & & \\
\pi_V^G R(2^\ell-2) \arrow[d] \arrow[r, phantom, "\text{follows from 7.13}"] & & \pi_V^u R(2^\ell-2) \arrow[ll, "\mathrm{Tr}"'] \arrow[d] \\
\pi_{|V|}^u R(2^\ell-2) & & \pi_{|V|}^u R(2^\ell-2) \arrow[ll, "(1-\gamma)"']
\end{tikzcd}$$

QED

Lemma 7.13 Let M be a G-spectrum which is ≥ 0, i.e. built of slice cells of dimension ≥ 0. Then

$$
\begin{tikzcd}
\pi_0^G C_{2+} \wedge M \arrow[r, dashed, "\text{onto}", "\beta"'] \arrow[d, phantom, "\parallel"] & \pi_0^G EC_{2+} \wedge M \arrow[r, "\alpha"] & \pi_0^G M \\
\pi_0^H M \arrow[rr] & & \pi_0^H M \arrow[u, "Tr"']
\end{tikzcd}
$$

The image of α is the same as that of transfer

Proof: β is onto because in dim 0 the 2 spectra are the same. QED

This completes the proof of 7.12 and hence of the Reduction Theorem.

Recap

Browder's Thm 1969 A framed manifold with nontrivial Kervaire invariant must have dimension $2^{j+1}-2$ for some $j \geq 1$. Such a manifold exists iff $h_j^2 \in \operatorname{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2,\mathbb{Z}/2)$ is a permanent cycle in the Adams SS.

An element in $\pi_{2^{j+1}-2} S^0$ representing h_j^2 (if it exists) is called θ_j. It was known to exist for $1 \leq j \leq 5$.

Thm (HHR 2009) θ_j does not exist for $j \geq 7$.

The case $j = 6$ is still open.

Strategy of proof: Construct a map $S^0 \longrightarrow \Omega$ where Ω is a spectrum with

a) Detection If $\exists\, \theta_j$, its image in $\pi_{2^{j+1}-2}\,\Omega$ is nontrivial

b) Periodicity $\Sigma^{256}\Omega \simeq \Omega$

c) Gap $\pi_k\,\Omega = 0$ for $-4 < k < 0$.

If $\exists\, \theta_7$, it has nontrivial in $\pi_{254}\,\Omega$ by a)

$$\begin{aligned} \pi_{254}\,\Omega &= \pi_{-2}\,\Omega && \text{by b)} \\ &= 0 && \text{by c)} \end{aligned}$$

Our Alpine hike:

* For each $n \geq 0$ and each prime p there is a spectrum E_n (Morava E-theory) related to height n formal group laws.
* There a profinite gp S_n (nth Morava stabilizer gp, the automorphism of a height n FGL over $\overline{\mathbb{F}_p}$) it "acts" on E_n. The "action" is only defined up to homotopy, but it is possible to construct homotopy fixed point sets E_n^{hG} for any closed subgp $G \subset S_n$.

* The case of finite is of particular interest. S_n contains a cyclic subgp of order p^{s+1} iff $(p-1)p^s \mid n$. We were studying some such E_n^{hG}. For $(p, n) = (2, 4)$ we have a cyclic subgp of order 8.
We knew $E_4^{hC_8}$ has detection and periodicity, but we could not prove a gap theorem. SUMMER 2008

An alternate approach

* C_2 acts on MU by complex conjugation. This makes MU a C_2-equivariant spectrum. This leads to a C_{2^k}-equiv structure on $N_2^{2^k} MU = MU^{(k)}$, e.g. $MU^{(4)}$ is a C_8-spectrum. A relative of $MU^{(4)}$ will be our substitute for E_4.
* An equiv over a cyclic 2-gp can be studied via its slice tower, which is an equiv analog of the Postnikov tower.

Classically, $P^n X$, the nth Postnikov section of X, is the spectrum obtained by kill all homotopy gps above dim n.

$$P_n X \longrightarrow X \longrightarrow P^n X$$

$P_n X$ = n-connected cover of X

$$P_n^n X \longrightarrow P^n X \longrightarrow P^{n-1} X$$

$P_n^n X$ = Eilenberg-MacLane spectrum which single nontrivial htpy gp, $\pi_n X$.

The Postnikov tower is

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n+1}X \arrow[r] & P^nX \arrow[r] & P^{n-1}X \arrow[r] & \cdots \\
& P^{n+1}_{n+1}X \arrow[u] & P^n_nX \arrow[u] & P^{n-1}_{n-1}X \arrow[u] &
\end{tikzcd}$$

$$\varprojlim P^nX = X \quad \text{and} \quad \varinjlim P^nX = *$$

An equiv analog:

For each $H \subset G$, $m \in \mathbb{Z}$, let

$$\hat{S}(m\rho_H) = G_+ \wedge_H S^{m\rho_H} \qquad h = |H|$$

$$\underset{u}{\simeq} \bigvee_{|G/H|} S^{mh}$$

This is a G-spectrum of "dimension mh"

$P^n X$ is the G-spectrum obtained from X by killing all equiv maps to X from $\hat{S}(m\rho_H)$ for $mh > n$ and from

$$\Sigma^{-1}\hat{S}(m\rho_H) \quad " \quad mh-1 > n.$$

This leads to a tower similar to the one above. Here $P^n_n X$ has rich $\pi^G_\star$. The tower leads to a SS with

$$E_2^{s,\tau} = \pi^G_{\tau-s} P^s_s X \quad \text{with } \tau \in RO(G)$$

$$\Longrightarrow \pi^G_{\tau-s} X$$

In the classical case $E_2^{s,t} \neq 0$ only for $t = 2s$, and the the SS collapses from E_2. In equiv case it is interesting.

<u>Slice Thm</u> Let $X = MU^{(g/2)}$. Then $P_s^0 X$ is

* contractible if s is odd
* $H\underline{\mathbb{Z}} \wedge W$ where W is a certain wedge of $S^{m\rho_H}$ for $m|h| = s$ and $h \geq 1$ when s is even

This is a "formal" consequence of the Reduction Theorem.

What is this good for?

We can calculate $\pi_\star^G \tilde{S}(m\rho_H) \wedge H\underline{\mathbb{Z}}$ with relative ease. Each $(\tilde{S}(m\rho_H) \wedge H\underline{\mathbb{Z}})^G$ has the gap property (i.e. $\pi_{-2} = 0$) for all $m \in \mathbb{Z}$ and all $h > 1$.

It follows for certain $x \in \pi^G_{m\rho_G} MU^{(G/2)}$,
$(x^{-1} MU^{(G/2)})^G$ also has the gap property.

We can also show that
$(x^{-1} MU^{(G/2)})^{hG}$ has periodicity

For appropriate G and x, $(x^{-1}MU^{(g/2)})^{hG}$ has detection property.

Fixed Point Theorem

$$(x^{-1}MU^{(g/2)})^{hG} = (x^{-1}MU^{(g/2)})^{G}$$

The simplest case with detection property is $G = C_8$, $x \in \pi_{19\rho_8}^{G} MU^{(4)}$;

periodicity dimension is 256.

Historical note: The odd primary case.

There is no known odd primary analog to any manifold theoretic statement. But there are odd primary analogs

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2,\mathbb{Z}/2) \qquad b_{j-1} \in \mathrm{Ext}_A^{2,2(p-1)p^j}(\mathbb{Z}/p,\mathbb{Z}/p) \qquad j \geq 1$$

$$\beta_{2^{j-1}/2^{j-1}} \qquad \text{Novikov SS} \qquad \beta_{p^{j-1}/p^{j-1}}$$

Thm (Toda 1967) For each $p>2$

$d_{2p-1}(b_1) = \alpha_1 b_0^p \qquad \alpha_1 \in \pi_{2p-3} S^0$

i.e. $\nexists\, \theta_2$. NO ANALOG at $p=2$.

Lemma There are relations ($\theta_j = b_{j-1}$ on $\beta_{p^{j-1}/p^{j-1}}$)

$$\theta_1^{p^j}\theta_{j+2} = \theta_2^{p^j}\theta_{j+1} \quad \text{for } j \geq 1$$

ALSO HOLD for $p=2$

Detection Thm 1978 For monomial θ^J in the $\theta_j = \beta_{p^{j-1}/p^{j-1}}$, θ^J and $\alpha_1\theta^J$ has nontrivial image (if they exist) in $\pi_*(E_{p-1}^{hC_p})$. Originally in terms of $H^*(C_p;\pi_* E_{p-1})$.

These three results imply

$d_{2p-1}(\beta_{p^{j-1}/p^{j-1}}) \neq 0$ for all $j > 1$

CAVEAT at $p=3$. The corresponding statemt in the Adams SS is false

$d_5(b_2) \overset{?}{=} \alpha_1 b_1^3$ $\quad$ dim $b_2 = 106$

but $\alpha_1 b_1^3 = 0$ in E_2

In Novikov SS

$\beta_{9/9}$ is not a perm cycle but

$\beta_{9/9} \pm \beta_7$ is a permanent cycle

i.e. $\theta_1 \theta_3$ exists at $p=3$

θ_2 does not exist.

We do not have an equivariant approach to this problem, i.e. we do not have a C_p-action on

$$MU^{(p-1)} \quad \text{for } p > 2.$$