MATH 549 4-22-10

Note Title 4/22/2010

Proof of Alice Theorem is in 2 steps

(1) Some formal arguments that reduce us to a special case called the Reduction Theorem

For the trivial gp it says if we kill all htpy groups, $\pi_* MU$ in positive dimensions, we get $H\mathbb{Z}$.
There is an equiv version which is not obvious.

(2) Prove the Reduction Thm. This requires some calculation.

Some formalities

Properties of the norm functor N_H^G

If $H \subset G$ and X is an H-equiv spectrum, then $X^{(g/h)}$ $\quad g = |G| \quad h = |H|$

has a G-action by permuting the factors

On the space level

$\mathrm{Map}_H(G, X)$ is a G-space

underlain is $X^{g/h}$.

Need a formula for $N_H^G(X_1 \vee X_2 \vee \cdots)$

Example $G = C_4$, $H = C_2$

$$X = S^0 \vee S^{\rho_2}$$

$$X^{(2)} = (S^0 \vee S^{\rho_2}) \wedge (S^0 \vee S^{\rho_2})$$

$$= (S^0 \wedge S^0) \vee (S^0 \wedge S^{\rho_2}) \vee (S^{\rho_2} \wedge S^0) \vee (S^{\rho_2} \wedge S^{\rho_2})$$

$$N_2^4 X = S^0 \vee (C_{4+} \wedge_{C_2} S^{\rho_2}) \vee S^{\rho_4}$$

[For a rep V of H, $N_H^G S^V = S^{\mathrm{Ind}_H^G V}$]

$$= N_2^4 S^0 \vee (\text{same}) \vee N_2^4 S^{\rho_2}$$

Let $X = \bigvee_{j \in J} X_j$. Each X_j is an H-spectrum.

Want to describe $N_H^G X$.

$K = \hom_{set}(G/H, J)$ is a G-set

For $k \in K$, let $H \subset G_k \subset G$ be its stabilizer gp

Let $[k]$ be the G-orbit of k. It is iso as a G-set to G/G_k.

$$X_{\bar{k}} = \bigwedge_{t \in G/G_k} X_{\bar{k}(t)}$$

$$\begin{tikzcd} G/H \arrow[r, "k"] \arrow[d] & J \\ G/G_k \arrow[ur, dotted, "\bar{k}"'] & \end{tikzcd}$$

$$X_k = N_H^{G_k} X_{\bar{k}}$$

Prop $N_H^G X = \bigvee_{[k] \in K/G} G_+ \wedge_{G_k} X_k$

$$X = \bigvee_{j \in J} X_j$$

Some equiv ring spectra underlain by wedges of spheres

Let $H < G$, V = rep of H

$Y = S^0[S^V] = \bigvee_{i \geq 0} S^{iV}$ is an H-spectrum with a multiplication

$$Y \wedge Y = \bigvee_{i,j \geq 0} S^{iV} \wedge S^{jV} \simeq \bigvee_{i,j \geq 0} S^{(i+j)V} \longrightarrow Y$$

This is associative but not comm. (?)

Let $\bar{x} \in \pi_V^H S^0[S^V]$ denote the inclusion

$$S^V \xrightarrow{\bar{x}} S^0[S^V] \; . \qquad S^0[\bar{x}] := S^0[S^V]$$

$N_H^G S^0[\bar{x}]$ is a known quantity

$$\| \qquad S^0[G \cdot \bar{x}]$$

Think of this as a "polynomial algebra" on one generator.

We can smash such things for various V's and get polynomial algebras on several generators:

Ex. Recall $\pi_*^u[MU_{\mathbb{R}}] = \mathbb{Z}[x_1, x_2, \ldots]$
$$\pi_* MU$$

For each x_i there is an $\bar{x}_i \in \pi_{i\rho_2}^{C_2} MU_{\mathbb{R}}$

which $i_0^* \bar{x}_i = x_i$

(i_0^* = forgetful functor or restriction to trivial gp)

This means that for each $i > 0$

there is a C_2-map $S^0[S^{i\rho_2}] \longrightarrow MU_{\mathbb{R}}$

representing all powers of $\bar{x}_1$

Let $S^0[\bar{x}_1] := S^0[S^{i\rho_2}]$.

Let $W = S^0[\bar{x}_1] \wedge S^0[\bar{x}_2] \wedge \cdots$

It is an associative ring spectrum

There is map $W \xrightarrow{f} MU_{\mathbb{R}}$

which is multiplicative refinement

of $\pi_*^u(MU_{\mathbb{R}}) = \pi_*(MU)$.

For $G = C_{2^k}$, then the map

$$A = N_2^{2^k} W \xrightarrow{N_2^{2^k} f} N_2^{2^k} MU_{\mathbb{R}} = MU^{(G/2)}$$

is a mult. refinement of $\pi_* MU^{(G/2)}$

More formal machinery

How to construct "ideals" in the ring spectrum.

Fix a gp G and let $\{H_i \subset G\}$ with V_i a rep of H_i. For each of these we get $S^0[\bar{x}_i]$ and

$$N_{H_i}^{G} S^0[\bar{x}_i] = S^0[G \cdot \bar{x}_i]$$

We can smash these together and get an "equiv polynomial algebras" T

How to construct ideals generated by sets of monomials

Let $\quad J = \coprod_i G/H_i$, a G-set

$\mathbb{N}_0 = \{0, 1, 2, 3 \cdots\}$

$\mathbb{N}_0^J$ = set of finitely supported functions $f: J \longrightarrow \mathbb{N}_0$

also a G-set.

$V_f = f(1)V_1 + f(2)V_2 + f(3)V_3 \cdots$

is a rep of the stabilizer gp G_f

Then $T = \bigvee_{f \in \mathbb{N}_0^J} S^{V_f}$

$$= \bigvee_{[f] \in \mathbb{N}_0^J / G} G_+ \wedge_{G_f} S^{V_f}$$

$\mathbb{N}_0^J$ is a commutative monoid

An ideal $I \subset \mathbb{N}_0^J$ is a subset with $I + \mathbb{N}_0^J \subset I$

For a G-invariant ideal I, let

$$T_I = \bigvee_{f \in I} S^{V_f}$$

This an equiv sub-bimodule of T

$$T \cap T_I \longrightarrow T_I$$

$$T_I \cap T \longrightarrow$$

<u>Example</u> Let $\dim : \mathbb{N}_0^J \longrightarrow \mathbb{N}_0$

$$\dim f = \dim V_f$$

$$= \sum_{j \in J} f(j) \dim V_j$$

Define an ideal $I_d \subset \mathbb{N}_0^J$

$$I_d = \{ f \in \mathbb{N}_0^J : \dim f \geq d \}$$

It is G-invariant

Let $M_d = T_{I_d} = \bigvee_{\dim V_f \geq d} S^{V_f}$

$$M_d / M_{d+1} = \bigvee_{\dim V_f = d} S^{V_f}$$

Example of interest

$G = C_2^k$, $H_i = C_2$ for $i = 1, 2, \ldots$

$V_i = i\rho_2$

$\overline{M}_i : S^{V_i} \longrightarrow MU_{\mathbb{R}}$ a certain element.

$A = S^0[G \cdot \bar{m}_1, G \cdot \bar{m}_2, \ldots] \longrightarrow N_2^g MU = MU^{(g/2)}$

is a mult. refinement of $\pi_*^u MU^{(g/2)}$

$A \supset M_{2d}$ described above

M_{2d}/M_{2d+2} = wedge of slice cells

Let $K_{2d} = MU^{(g/2)} \wedge_A M_{2d}$

Note $MU^{(g/2)}$ and M_{2d} are both A-modules

$MU^{(g/2)} = K_0 \supset K_2 \supset K_4 \supset K_6 \ldots$

Let $\tilde{P}^{2d} MU^{(g/2)} = MU^{g/2} / K_{2d+2}$

$$\tilde{P}^{0} MU^{(g/2)} = K_0 / K_2$$

Reduction Thm

$$R(\infty) := K_0 / K_2 = H\underline{\mathbb{Z}}$$

Hard to prove

Formal fact

$$K_{2d} / K_{2d+2} = R(\infty) \wedge \left(M_{2d} / M_{2d+2} \right)$$

The layers in our new tower

$\{\tilde{P}^{2d} = MU^{((G))}/K_{2d+2}\}$ and

as above. If we knew $R(\infty) = 4\mathbb{Z}$,

then we could identify $\{\tilde{P}^{d}\}$

with the slice tower.