MATH 549 4-3-10

Note Title 4/2/2010

Last time: $G = C_2$

Studied the slice SS for $MU_{\mathbb{R}}$ and $X = \Sigma_1^{-1} MU_{\mathbb{R}}$ and found

① X^G has gap property, $\pi_{-2} X^G = 0$

② X^{hG} has periodicity, $\Sigma^8 X^{hG} \simeq X^{hG}$

Want to know $X^G \simeq X^{hG}$

Key fact $X \wedge \tilde{E}G = *$

We know $\pi_\star^G (X \wedge \tilde{E}G) = a^{-1} \pi_\star X$ and in $\pi_\star X$, $a^3 = 0$.

Thm Let $G = C_{2^n}$ and X is a G-ring-spectrum s.t. for each subg $\underline{H \leq G}$ $(H \neq \{e\})$ [let $H' \subset H$ be the subgp of index 2 and $H_2 = H/H'$], $\Phi^H X$ is contractible. Then $X^G \simeq X^{hG}$.

Recall the isotropy separation sequence for H
$$EH_{2+} \longrightarrow S^0 \longrightarrow \widetilde{E}H_2$$
$$\Phi^H X := (\widetilde{E}H_2 \wedge X)^H$$

Remark EG_2 is a G-space with

$$(EG_2)^H = \begin{cases} \text{empty for } H = G \\ \text{contractible for } H \neq G \end{cases}$$

Proof We have an equiv $EG_+ \longrightarrow S^0$

Mapping both to X gives a map

$$F(S^0, X) = X \xrightarrow{\varphi} F(EG_+, X)$$

Apply H-fixed pts gives

$$X^H \xrightarrow{\varphi^H} X^{hH}$$

Will show φ^H is an equiv by induction on H. $\varphi^{\{e\}}$ is identity map on X.

For the inductive step, will smash φ with

$$EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G$$

and get a diagram

$$\begin{tikzcd}
EG_+ \wedge X \arrow[r] \arrow[d, "\varphi'", "\simeq_G"'] & X \arrow[r] \arrow[d, "\varphi"] & \tilde{E}G \wedge X \arrow[d, "\varphi''"] \\
EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \tilde{E}G \wedge F(EG_+, X)
\end{tikzcd}$$

φ' is an G-equiv because the underlying map $\varphi : X \to F(EG_+, X)$ is a nonequiv equivalence and EG is built out of free G-cells. Suffices to show φ'' is a G-equiv.

Will show that source and target of φ'' are G-contractible. Recall a G-spectrum Y is G-contractible if $Y^H \simeq *$ for each subgp $H \subset G$.

To study φ'' we will smash it with the ISS for $G \supseteq H \neq \{e\}$

$$EH_{2+} \longrightarrow S^0 \longrightarrow \tilde{E}H_2$$

$$\tilde{E}G \wedge X \xrightarrow{\varphi''} \tilde{E}G \wedge F(EG_+, X)$$

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We know $\underline{\tilde{E}H_2 \wedge \tilde{E}G \wedge X}$ is contractible over H' (since $\tilde{E}H_2$ is) so it suffices to show its H-fixed pt spectrum is contractible. It is

$$\Phi^H(\tilde{EG} \wedge X) = \Phi^H(\tilde{EG}) \wedge \Phi^H(X)$$

and $\Phi^H(X) \simeq *$ by hypothesis.

↓ forgot to say:

assume X is a ring spectrum

$F(EG_+, X)$ is an X-module

in ① it suffices to deal with $\tilde{EG} \wedge X$.

We need to show

$$E\mathcal{H}_{2+} \wedge \widetilde{EG} \wedge X \text{ is } H\text{-contractible}$$

Because $E\mathcal{H}_{2+}$ is built out of free H_2-cells, it suffices to show

$$\widetilde{EG} \wedge X \text{ is } H'\text{-contractible.}$$

We know this by induction.

QED

Next: Describe the likely slices of $MU^{(2)}$ as a C_4-spectrum and $MU^{(4)}$ as a C_8-spectrum.

Recall there is a big theorem (yet to be proved but very plausible) identifying the slices. In the MU case it says the odd slices are contractible and the $2m$th slice is a certain wedge of

$$S^{m\rho} \wedge H\underline{\mathbb{Z}}.$$

What is $H_* MU^{(4)}$?

$$H_*(MU) = \mathbb{Z}[b_1, b_2, \dots] \qquad b_i \in H_{2i}$$

There are 4 ways to map $MU \to MU^{(4)}$

$$MU = S^0 \wedge S^0 \wedge S^0 \wedge MU \longrightarrow MU \wedge MU \wedge MU \wedge MU$$

$$S^0 \wedge S^0 \wedge MU \wedge S^0 \longrightarrow$$

etc.

They are related by the action of C_8

Let $b_i(j)$ for $1 \le j \le 4$ be the 4 images of b_i in $H_* MU^{(4)}$

If γ is a gen of C_8

$$\gamma\left(b_i(j)\right) = \begin{cases} b_i(j+1) & \text{for } 1 \le j \le 3 \\ (-1)^i\, b_i(1) & j = 4 \end{cases}$$

$\overline{\pi}_*^u MU^{(4)}$ has a similar description

$$\pi_*^u MU^{(4)} = \mathbb{Z}\left[m_i(j) \; : \; i > 0, \; 1 \le j \le 4 \right] \qquad m_i(j) \in \pi_{2i}$$

The action of C_8 is similar.

Recall for a subgp $H \subset G = C_{2^n}$, let ρ_H denote its regular rep

$$\hat{S}(m\rho_H) = G_+ \wedge_H S^{m\rho_H}$$

Def Suppose X is a G-spectrum with $\pi_k^u(X)$ is free abelian. A <u>refinement</u> of $\pi_k^u(X)$ is a map

$$\hat{W} \longrightarrow X \quad \text{equivariant}$$

where $\hat{W}$ is a wedge of $\hat{S}(m\rho_h)$ with $mh = k$ whose underlying spheres represent all the generators of $\pi_k^u(X)$.

Examples $\pi_2^u MU^{(4)} = \mathbb{Z}^4$

$= \mathbb{Z}\{m_1(j) : 1 \le j \le 4\}$

$\hat{S}(\rho_2) = C_{8+} \wedge_{C_2} S^{\rho_2}$

There is map $\hat{S}(\rho_2) \longrightarrow MU^{(4)}$

$\pi_4^u MU^{(4)}$ has 14 generators which are permuted up to sign by C_8.

The orbits

$\hat{S}(2\rho_2)$	$\{m_1(1)^2, m_1(2)^2, m_1(3)^2, m_1(4)^2\}$	$\mathbb{Z}^4$
$\hat{S}(2\rho_2)$	$\{m_1(1)m_1(2), m_1(2)m_1(3), m_1(3)m_1(4), m_1(4)m_1(1)\}$	$\mathbb{Z}^4$

$\hat{S}(2p_2)$ $\{ M_2(1), M_2(2), M_2(3), M_2(4) \}$

$\hat{S}(p_4)$ $\{ M_1(1)\, M_1(3),\ M_1(2)\, M_1(4) \}$

$$\hat{W} = \hat{S}(2p_2) \vee \hat{S}(2p_2) \vee \hat{S}(2p_2) \vee \hat{S}(p_4)$$

We can make similar calculations in higher dims

1. We will never get an 8-element orbit.
2. The first 1-element orbit is in dim 8, namely

S^{P_8} $\{ M_1(1)\ M_1(2)\ M_1(3)\ M_1(4) \}$

In dim $2k$ we get a $\hat{W}_k$ and the corresponding slice is

$$\hat{W}_k \wedge H\underline{\mathbb{Z}}.$$

We need to look at

$$\pi_* \ \hat{S}(m\rho_4) \wedge H\underline{\mathbb{Z}}.$$

In $MU_{\mathbb{R}}$ $\pi_*^u MU = \mathbb{Z}[x_1, x_2, x_3 \cdots]$

x_i is refined by $S^{i\rho}$

ρ means ρ_2 = reg rep of C_2

x_1^2 is refined by a map from

$S^\rho \wedge S^\rho = S^{2\rho}$