

MATH 549 4-19-10

Note Title

4/19/2010

Toward the Detection Theorem

We need the Adams-Novikov SS

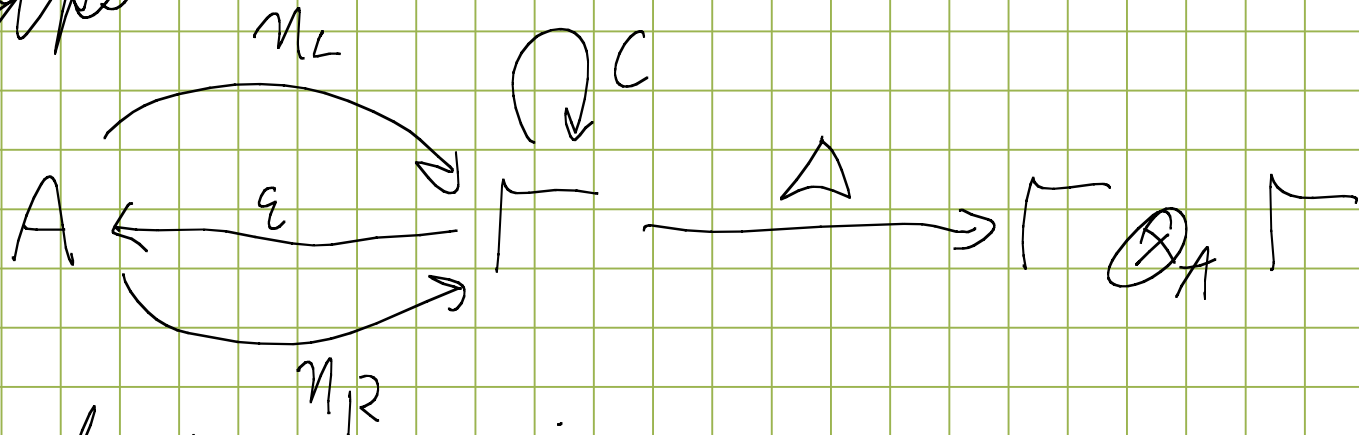
Recall, for a "nice" ring spectrum R
the pair

$$(A, \Gamma) = (\pi_* R, \pi_* R \wedge R)$$

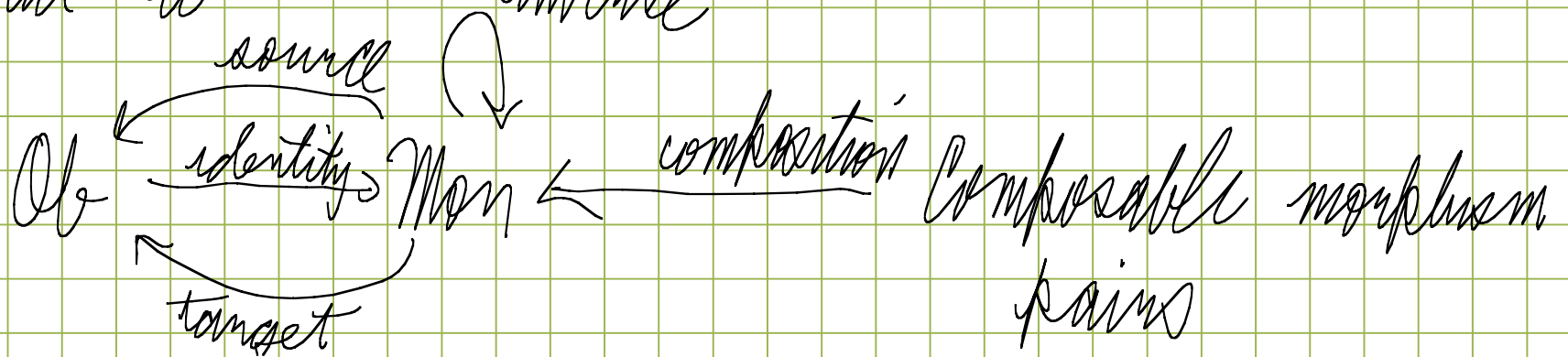
is a Hopf algebra over $K = \pi_0 R$, i.e.

it is a cogroupoid ^{object} in the category of
graded comm K -algebras. This means
for any such algebra S , the sets

$\text{Hom}_{K\text{-alg}}(A, S)$ and $\text{Hom}_{K\text{-alg}}(\Gamma, S)$
 are the object + morphism sets of
 a groupoid. There are 3 structure
 maps



dual to inverse



The E_2 -term of the R -based Adams SS for X is

$$\text{Ext}_R(A, R_*(X)) =: \text{Ext}(R_*(X))$$

Two examples

1) Classical case $R = H/\mathbb{Z}$

$$\pi_* R = \pi_0 R = K = \mathbb{Z}/2$$

$$\pi_* R \cap R = A_* = \text{dual Steenrod alg}$$

$\pi_R = \pi_L$ and we get a Hopf algebra over $\mathbb{Z}/2$

2) $R = MU$, $\pi_0 R = \mathbb{Z}$

$$|x_i| = |b_i| = 2i$$

$$A = \pi_* MU = \mathbb{Z}[x_1, x_2, \dots] = \text{Lazard ring} = L$$

$$\text{Hom}_{\mathbb{Z}\text{-alg}}(A, S) = \text{set of FGAs } / S$$

$$\Gamma = \pi_* MU \wedge MU = MU_* (MU) = L[b_1, b_2, \dots] = LB$$

$$\text{Hom}_{\mathbb{Z}\text{-alg}}(LB, S') = \left\{ \begin{array}{l} \text{pairs of FGAs } F \text{ and } F' \\ \text{over } S \text{ with a strict} \\ \text{iso } f(x) = x + b_1 x^2 + b_2 x^3 + \dots \end{array} \right.$$

$$f(F(x, y)) = F'(f(x), f(y)) \quad \left. \vphantom{f(F(x, y))} \right\}$$

The groupoid is that of FGAs + strict isomorphisms between them.

To describe the structure maps, we tensor with \mathbb{Q}

$$\log x = \sum m_i x^{i+1} \quad \text{where} \quad m_i = \frac{[\mathbb{C}P^i]}{i+1}$$
$$e(L \otimes \mathbb{Q})[[x]] \quad m_0 = 1$$

$c(b_i)$ is defined recursively by

$$\sum_{i \geq 0} c(b_i) \left(\sum_{j \geq 0} b_j \right)^{i+1} = 1 \quad \text{where} \quad b_0 = 1$$

follows

$$f^{-1}(f(x)) = x$$

$$f^{-1}(x) = \sum c(b_i) x^{i+1}$$

$$f(x) = \sum b_i x^{i+1}$$

Remark 1) For the Steenrod algebra

Consider $f(x) = \sum_{i \geq 0} \xi_i x^{2^i}$ $\xi_0 = 1$

$$f^{-1}(x) = \sum_{i \geq 0} c(\xi_i) x^{2^i}$$

Can also determine

the Milnor isoproduct this way.

Remark 2) $c(b_i) \in \mathbb{Z}[b_1, b_2, \dots]$

We do not need more of $\pi_* MV$ here

This related to the fact for each \mathbb{Z} -algebra S the groupoid is split

The group is $\text{Hom}(B, S)$

where $B = \mathbb{Z}[b_1, b_2, \dots]$
 equipped the conjugation c defined
 above.

Back to our structure:

$\eta_L: L \rightarrow LB$ standard inclusion

$\varepsilon: LB \rightarrow L$ $b_i \mapsto 0$ for $i > 0$

η_R defined over \mathbb{Q} by

$$\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i \left(\sum_{j \geq 0} c(b_j) \right)^{i+1}$$

Δ defined by

$$\sum_{i \geq 0} \Delta(b_i) = \sum_{j \geq 0} \left(\sum_{i \geq 0} b_i \right)^{j+1} \otimes b_j$$

$C(m_i) = \mathcal{N}_R(m_i)$ as defined above

The relation between the m_i and the x_i is complicated.

For computational purposes, localize at a prime p , and replace MV by BP, the Brown-Peterson spectrum.

Def A FGL over a torsion free ring R is p -typical if its logarithm has the form $\log x = \sum_{i \geq 0} l_i x^{p^i}$, $l_0 = 1$

Thm (Cartier 1960s) Any FGL F over
a $\mathbb{Z}_{(p)}$ -algebra, is canonically iso
to a - typical one G .

$$\Downarrow \log_{\sqrt{F}} \chi = \sum_{i \geq 0} m_i \chi^{i+1}$$

$$\text{then } \log_{\sqrt{G}} \chi = \sum_{i \geq 0} m_{p^i-1} \chi^{p^i}$$

Thm (Brown-Peterson 1967)

$$MU_{(p)} = \bigvee \Sigma^? BP$$

$$\pi_* BP = \mathbb{Z}_{(p)} [v_1, v_2, \dots] \quad |v_i| = 2(p^i - 1)$$

Thm (Quillen 1969) BP is a mil
ring spectrum with the following
structure

$$BP_* \otimes \mathbb{Q} = \mathbb{Q}[l_1, l_2, l_3, \dots]$$

$$|l_i| = |v_i| = |t_i| = 2(p^i - 1)$$

$$\Gamma = BP_*(BP) = BP_*[t_1, t_2, \dots]$$

$$\eta_R(l_n) = \sum_{0 \leq i \leq n} l_i t_{n-1}^{p^i} \quad l_0 = t_0 = 1$$

(more explicit than $\eta_R(m_n)$)

Δ is determined by

$$\sum_{i \geq j \geq 0} l_i \Delta(t_j)^{p^i} = \sum_{i \geq j \geq k \geq 0} l_i t_j^{p^i} \otimes t_k^{p^i + j}$$

Remark This Hopf algebra is not split, unlike $MU_* (MU)$.

We have the Adams-Novikov SS based on BP-theory.

The relation between the v_i and h_i is not complicated. One choice of the v_i due to Hazewinkel is given recursively by

$$p h_m = \sum_{0 \leq i < m} h_i v_{m-1}^{p^i} \quad h_0 = 1$$

This leads to

$$l_1 = v_1 / p$$

$$l_2 = v_2 / p + v_1^{1+p} / p^2$$

$$l_3 = v_3 / p + (v_1 v_2^p + v_2 v_1^{p^2}) / p^2 + v_1^{1+p+p^2} / p^3$$

etc

We need to know $\eta_R(v_i)$

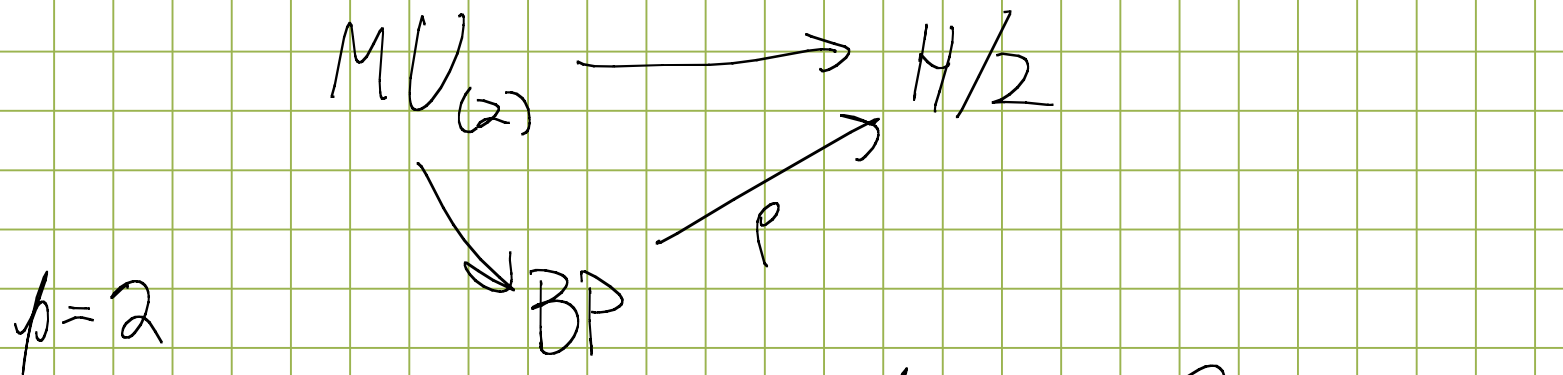
We have nice formula for $\eta_R(l_i)$

$$\eta_R(v_1) = v_1 + pt_1$$

$$\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1 \pmod{p}$$

There is a relation between the
Novikov and Adams Ext groups

One has maps of ring spectra



p induces a map of Ext groups

h_2^2 is in the image of p

h_0

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only

for $1 \leq j \leq 3$.
(This leads to a proof that h_j for $j > 3$ is not a permanent cycle.)

Extra meetings

Thursday 4/22

3:30

Friday 4/20

4:00