MATH 549 4-19-10

Note Title 4/19/2010

Toward the Detection Theorem

We need the Adams-Novikov SS

Recall, for a "nice" ring spectrum R the pair

$$(A, \Gamma) = (\pi_* R, \pi_* R \wedge R)$$

is a Hopf algebroid over $K = \pi_0 R$, i.e. it is a cogroupoid object in the category of graded comm K-algebras. This means for any such algebra S, the sets

$\mathrm{Hom}_{K\text{-alg}}(A, S)$ and $\mathrm{Hom}_{K\text{-alg}}(\Gamma, S)$ are the object + morphism sets of a groupoid. There are 5 structure maps

$$\begin{tikzcd}
A \arrow[r, bend left, "\eta_L"] \arrow[r, bend right, "\eta_R"'] & \Gamma \arrow[l, "\varepsilon"'] \arrow[loop above, "c"] \arrow[r, "\Delta"] & \Gamma \otimes_A \Gamma
\end{tikzcd}$$

dual to

$$\begin{tikzcd}
\mathrm{Ob} \arrow[r, "\text{identity}"] & \mathrm{Mor} \arrow[l, bend right, "\text{source}"'] \arrow[l, bend left, "\text{target}"] \arrow[loop above, "\text{inverse}"] & \text{Composable morphism pairs} \arrow[l, "\text{composition}"']
\end{tikzcd}$$

The E_2-term of the R-based Adams SS for X is

$$\mathrm{Ext}_\Gamma(A, R_*(X)) =: \mathrm{Ext}(R_*(X))$$

Two examples

1) Classical case $R = H/2$

$\pi_* R = \pi_0 R = K = \mathbb{Z}/2$

$\pi_* R \wedge R = A_* =$ dual Steenrod alg

$\eta_R = \eta_L$ and we get a Hopf algebra over $\mathbb{Z}/2$

$|x_i| = 1 \quad |b_i| = 2i$

2) $R = MU$, $\pi_0 R = \mathbb{Z}$

$A = \pi_* MU = \mathbb{Z}[x_1, x_2 \cdots] =$ Lazard ring $= L$

$\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(A, S) =$ set of FGLs / S

$\Gamma = \pi_* MU \wedge MU = MU_*(MU) = L[b_1, b_2 \cdots] = LB$

$\mathrm{Hom}_{\mathbb{Z}\text{-alg}}(LB, S) = \{$ pairs of FGL's F and F' over S with a strict iso

$$f(x) = x + b_1 x^2 + b_2 x^3 + \cdots$$

$$f(F(x,y)) = F'(f(x), f(y)) \}$$

The groupoid is that of FGLs + strict isomorphisms between them.

To describe the structure maps, we tensor with $\mathbb{Q}$

$\log x = \sum m_i x^{i+1}$ where $m_i = \dfrac{[\mathbb{C}P^i]}{i+1}$

$\in (L \otimes \mathbb{Q})[[x]]$ $\qquad m_0 = 1$

$c(b_i)$ is defined recursively by

$$\sum_{i \geq 0} c(b_i) \Big(\sum_{j \geq 0} b_j \Big)^{i+1} = 1 \quad \text{where } b_0 = 1$$

follows $\qquad f^{-1}(f(x)) = x$

$f^{-1}(x) = \sum c(b_i) x^{i+1}$

$f(x) = \sum b_i x^{i+1}$

Remark 1) For the Steenrod algebra

Consider $f(x) = \sum_{i=0} \xi_i x^{2^i}$ $\xi_0 = 1$

$$f^{-1}(x) = \sum_{i=0} c(\xi_i) x^{2^i}$$

Can also derived the Milnor coproduct this way.

Remark 2) $c(b_i) \in \mathbb{Z}[b_1, b_2 \cdots]$

We do not need more of $\pi_* MU$ here

This related to the fact for each $\mathbb{Z}$-algebra S the groupoid is split

The group is $\mathrm{Hom}(B, S)$

where $B = \mathbb{Z}[b_1, b_2, \dots]$
equipped the conjugation c defined above.

Back to our structure:

$\eta_L : L \longrightarrow LB$ standard inclusion

$\varepsilon : LB \longrightarrow L$ $\quad b_i \longmapsto 0$ for $i > 0$

η_R defined over $\mathbb{Q}$ by

$$\sum_{i\ge 0} \eta_R(m_i) = \sum_{i\ge 0} m_i \Big(\sum_{j\ge 0} c(b_j)\Big)^{i+1}$$

Δ defined by

$$\sum_{i\ge 0} \Delta(b_i) = \sum_{j\ge 0} \Big(\sum_{i\ge 0} b_i\Big)^{j+1} \otimes b_j$$

$C(m_i) = n_R(m_i)$ as defined above

The relation between the m_i and the x_i is complicated.

For computational purposes, localize at a prime p, and replace MU by BP, the Brown-Peterson spectrum.

<u>Def</u> A FGL over a torsion free ring R is <u>p-typical</u> if its logarithm has the form $\log x = \sum_{i \geq 0} l_i x^{p^i}$, $l_0 = 1$

Thm (Cartier 1960s) Any FGL F over a $\mathbb{Z}_{(p)}$-algebra is canonically iso to a p-typical one G.

If $\log_F x = \sum_{i\geq 0} m_i x^{i+1}$

then $\log_G x = \sum_{i\geq 0} m_{p^i-1} x^{p^i}$.

Thm (Brown-Peterson 1967)

$$MU_{(p)} = \bigvee \Sigma^{?} BP$$

$$\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2 \cdots] \qquad |v_i| = 2(p^i - 1)$$

Thm (Quillen 1969) BP is a mult ring spectrum with the following structure

$$BP_* \otimes \mathbb{Q} = \mathbb{Q}[l_1, l_2, l_3, \dots]$$

$$|l_i| = |v_i| = |t_i| = 2(p^i - 1)$$

$$\Gamma = BP_*(BP) = BP_*[t_1, t_2, \dots]$$

$$\eta_R(l_n) = \sum_{0 \le i \le n} l_i \, t_{n-i}^{p^i} \qquad l_0 = t_0 = 1$$

(more explicit than $\eta_R(m_n)$)

Δ is determined by

$$\sum_{i,j \ge 0} l_i \, \Delta(t_j)^{p^i} = \sum_{i,j,k \ge 0} l_i \, t_j^{p^i} \otimes t_k^{p^{i+j}}.$$

Remark This Hopf algebroid is not split, unlike $MU_*(MU)$.

We have the Adams-Novikov SS based on BP-theory.

The relation between the v_i and l_i is not complicated. One choice of the v_i due to Hazewinkel is given recursively by

$$p\, l_n = \sum_{0 \le i < n} l_i v_{n-i}^{p^i} \qquad l_0 = 1$$

This leads to

$$l_1 = v_1/p$$

$$l_2 = v_2/p + v_1^{1+p}/p^2$$

$$l_3 = v_3/p + (v_1 v_2^p + v_2 v_1^{p^2})/p^2 + v_1^{1+p+p^2}/p^3$$

etc

We need to know $\eta_R(v_i)$

We have nice formula for $\eta_R(l_i)$

$$\eta_R(v_1) = v_1 + pt_1$$

$$\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1 \qquad \mathrm{mod}\ p$$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1$$

$$\Delta(t_2) \equiv t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 \mod (v_1)$$

$$= \quad '' \quad \pm v_1 \sum_{0<i<p} p^{-1}\binom{p}{i} t_1^i \otimes t_1^{p-i}$$

We will need to know certain elements in $\mathrm{Ext}^1(BP_*)$ and $\mathrm{Ext}^2(BP_*)$

θ_j is related to $\mathrm{Ext}^{2, 2^{j+1}} BP_*$

Before we said θ_j was related to

$$h_j^2 \in \mathrm{Ext}_A^{2, 2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

There is a relation between the Novikov and Adams Ext groups

One has maps of ring spectra

$$\begin{tikzcd} MU_{(2)} \arrow[r] \arrow[d] & H/2 \\ BP \arrow[ur, "\rho"'] & \end{tikzcd}$$

$p=2$

ρ induces a map of Ext gps

h_j^2 is in the image of ρ

11

h_j only

for $1 \le j \le 3$.

(This leads to a proof that h_j for $j > 3$ is not a permanent cycle.)

Extra meetings

Thursday 4/22 3:30

Friday 4/30 4:00