MATH 549 3-31-10

Note Title 3/31/2010

Extra meeting 4:00 Friday 4/2/10
Room TBA

Thm $\pi_\star^{C_2}(\tilde{E}C_2 \wedge X) = a_\sigma^{-1}\pi_\star^{C_2}(X)$

<u>Proof</u> $\tilde{E}C_2 = S^{\infty\sigma} = \lim_{n\to\infty} S^{n\sigma}$

$$EC_{2+} \longrightarrow S^0 \longrightarrow \tilde{E}C_2$$

$$\begin{array}{ccccc} EC_{2+} & \longrightarrow & S^0 & \longrightarrow & \tilde{E}C_2 \\ \| & & & & \\ S^{\infty\sigma}_+ & \longrightarrow & S^0 & \longrightarrow & \Sigma S^{\infty\sigma} \\ & & & & \| \\ & & & & \lim_{n\to\infty} S^{1+n\sigma} \end{array}$$

In the colimit we have maps

$$\begin{array}{ccc} S^{1+n\sigma} & \longrightarrow & S^{1+(n+1)\sigma} \\ & & \| \\ & & S^{1+n\sigma} \wedge \left(S^0 \xrightarrow{a} S^\sigma\right) \end{array}$$

Passing to the limit is the same as inverting a.

QED

<u>Slice Theorem</u> The slices of $MU_{\mathbb{R}}$ are as described previously

This is plausible but hard to prove.

Will explore its computational consequences

Recall the slice SS for MU

$$E_2 = \mathbb{Z}[u, a, \bar{x}_1, \bar{x}_2, \ldots]$$

$$\left.\begin{array}{l} u_{2\sigma} = u \in E_2^{0, 2-2\sigma} \\ a_\sigma = a \in E_2^{1, 1-\sigma} \\ \qquad \bar{x}_i \in E_2^{0, i(1+\sigma)} \end{array}\right\} \text{permanent cycles}$$

In $a^{-1}\pi_\star^{C_2} MU$, $\bar{x}_1, \bar{x}_3, \bar{x}_7$ etc must vanish

$$\begin{array}{c} a^{-1}\pi_\star^{C_2} MU \\ \| \\ \pi_*(MO) \otimes \mathbb{Z}/2[a, a^{-1}] \end{array}$$

This means we have differentials

$$u^{2^k} \xrightarrow{d_{2^{k+2}-1}} a^{2^{k+2}-1}\,\bar{x}_{2^{k+1}-1} \quad \text{for } k \geq 0$$

$$u \xrightarrow{d_3} a^3 \bar{x}_1 \qquad u^3 \xrightarrow{d_3} a^3 x_1 u^2$$

$$u^2 \xrightarrow{d_7} a^7 \bar{x}_3$$

etc.

Strategy of Main Thm proof.

Construct a spectrum Ω with

i) DETECTION θ_j has nonzero image in $\pi_* \Omega$ if $\exists \theta_j$

ii) PERIODICITY $\Sigma^{256}\Omega = \Omega$

iii) GAP $\pi_{-2}\Omega = 0$

From MU we can get a spectrum with ii) and iii) but not i).

In MU we will $\bar{x}_1$

$$S^p \longrightarrow MU$$

$$S^p \wedge MU \longrightarrow MU \wedge MU \longrightarrow MU$$

$$MU \longrightarrow S^{-p} \wedge MU \longrightarrow S^{-2p} M \longrightarrow \cdots$$

gives $\bar{x}_1^{-1} MU$

In our SS

$$u \xrightarrow{d_3} a^3 \bar{x}_1$$

$$\bar{x}_1^{-1} u \xrightarrow{d_3} a^3$$

REMARK: In $\bar{x}_1^{-1} MU$ we get slices of the form $S^{-k\rho} \wedge H\underline{\mathbb{Z}}$ for $k \geq 0$. We have not computed $\pi_\star(S^{-k\rho} \wedge H\underline{\mathbb{Z}})$

$$u^2 \xrightarrow{d_7} a^7 \bar{x}_3 = 0 \text{ in } E_4$$

$$u \bar{x}_3 \bar{x}_1^{-1} a^4 \xrightarrow{d_3} a^7 \bar{x}_3$$

This means u^2 is a permanent cycle.

This means

$$E_\infty = \mathbb{Z}[2u, u^2, \bar{x}_1^{\pm}, \bar{x}_2, \bar{x}_3, \cdots, a]/(2a, a^3)$$

$$\cong \pi_\star^{C_2}(\bar{x}_1^{-1} MU)$$

For simplicity, ignore $\bar{x}_i$ for $i > 1$

Call the resulting spectrum $K_{\mathbb{R}}$

Recall $x_1^{-1} MU$ has K as a retract

(Conner-Floyd)

$$\pi_{\bigstar}^{C_2}(K_{\mathbb{R}}) = \mathbb{Z}[u^2, 2u, \bar{x}_1^{\pm}, a] \,/\, (2a, a^3)$$

Note $u^2 \in \pi^G_{4-4\sigma}$ $\quad \bar{x}_1^4 \in \pi^G_{4+4\sigma}$ $\quad G = C_2$

Let $\Delta = u^2 \bar{x}_1^4 \in \pi^G_8$

We can use this to make a self-map

$$\Sigma^8 K_{\mathbb{R}} \longrightarrow K_{\mathbb{R}}$$

Prop This is an ordinary equiv.

$$u^2 : S^4 \longrightarrow S^{4\sigma} \wedge H\mathbb{Z}$$

Ignoring the C_2 action we get

$$S^4 \longrightarrow S^4 \wedge H\mathbb{Z}$$

i.e. "u^2 restricts to 1"

Hence $u^2 \bar{x}_1^4$ restricts to an invertible map (QED)

$$\Sigma^8 K_{\mathbb{R}} \xrightarrow{\Delta} K_{\mathbb{R}}$$

Lemma Let $Y \xrightarrow{f} X$ be an equiv map which restricts (over the trivial gp) to a htpy equiv. Then it induces an equiv $Y^{hG} \longrightarrow X^{hG}$

<u>Proof</u>: $X^{hG} = F(EG, X)^G$

Consider the class of all equiv W s.t. f induces an equiv

$$F(W, \Sigma^? X) \longrightarrow F(W, X)$$

It is closed under suspensions + mapping cones, and includes G_+.

$$F(G_+, X)^G = X$$

Hence G_+ is such a W

$\Sigma^n G$ "

EG_+ is built out of free G-cells, it is also such a W.

QED

This is a periodicity thm

$$\Sigma^8 K_{\mathbb{R}}^{hC_2} \simeq K_{\mathbb{R}}^{hC_2}$$

Will also show $K_{\mathbb{R}}^{\mu C_2} \simeq K_{\mathbb{R}}^{C_2}$ later.

Will show $\pi_{-2}(K_{\mathbb{R}}^{C_2}) = 0$

GAP THEOREM.

The slices we get are $S^{k\rho} \wedge H\underline{\mathbb{Z}}$ for each $k \in \mathbb{Z}$. We need to compute $\pi_*(S^{k\rho} \wedge H\underline{\mathbb{Z}})$ for $k < 0$.

$S^{-k\rho}$ is the Spanier-Whitehead

dual of S^{kP}. This means:

for $k>0$, let $C(k)$ be the chain cx for S^{kP}

$$(*) \quad \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[C_2] \xleftarrow{1-\gamma} \mathbb{Z}[C_2] \xleftarrow{1+\gamma} \cdots$$

$$k \qquad k+1 \qquad 2k$$

$$\pi_*(S^{kP} \wedge H\mathbb{Z}) = H_*\left(\mathrm{Hom}_{\mathbb{Z}[C_2]}(\mathbb{Z}, C(k))\right)$$

$$\pi_{-i}(S^{-kP} \wedge H\mathbb{Z}) = H^i\left(\mathrm{Hom}_{\mathbb{Z}[C_2]}(C(k), \mathbb{Z})\right)$$

Apply $\mathrm{Hom}_{\mathbb{Z}[C_2]}(-, \mathbb{Z})$ to $(*)$

$$\underset{k}{\mathbb{Z}} \xrightarrow{1} \underset{k+1}{\mathbb{Z}} \xrightarrow{0} \underset{k+2}{\mathbb{Z}} \xrightarrow{2} \underset{k+3}{\mathbb{Z}} \xrightarrow{0} \underset{k+4}{\mathbb{Z}} \xrightarrow{2} \underset{k+5}{\mathbb{Z}} \cdots$$

$$\underbrace{0 \qquad 0}_{\text{GAP}} \qquad 0 \qquad \mathbb{Z}/2 \qquad 0 \qquad \mathbb{Z}/2 \cdots$$

$k=1$

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z}$$

$k=2$

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$k=3$

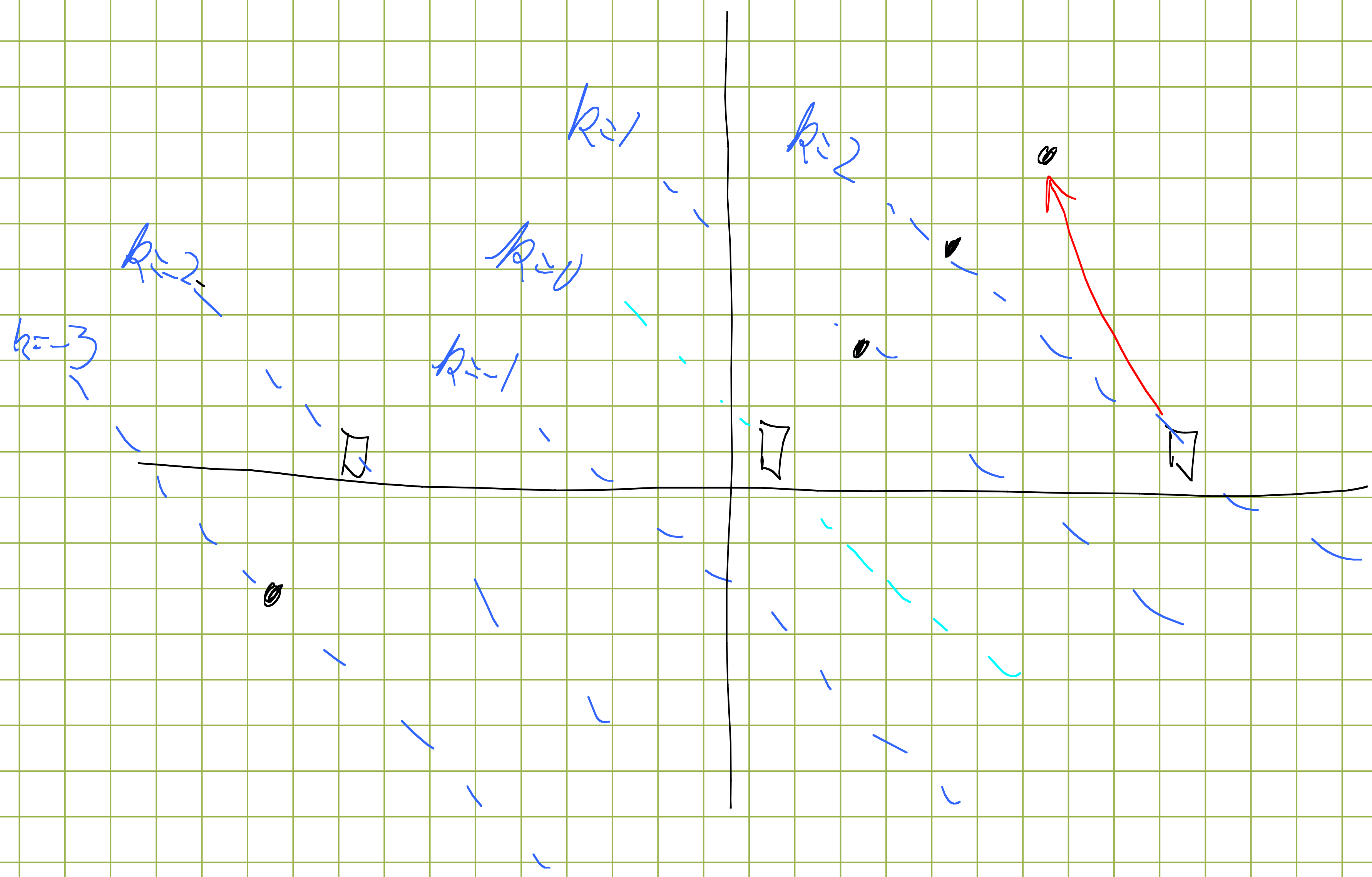
$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

$k=1$ $\quad \pi_*(S^{-\rho} \wedge H\mathbb{Z}) = 0$

$k=2$ $\quad \pi_i(S^{-2\rho} \wedge H\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=-4 \\ 0 & \text{else} \end{cases}$

$k=3 \qquad \pi_i(S^{-3\rho}, H\mathbb{Z}) = \begin{cases} \mathbb{Z}/2 & i=-6 \\ 0 & \text{else} \end{cases}$

This SS was studied by Dugger.


k=-3
k=-2
k=-1
k=0
k=1
k=2