MATH 549 3-29-10

Note Title 3/29/2010

Want to describe the slice SS for $MU_{\mathbb{R}}$ with $G = C_2$

Each slice is a wedge of copies of $S^{k\rho} \wedge H\underline{\mathbb{Z}}$

ρ = regular rep = $1 + \sigma$

σ = sign rep

How to compute $\pi_*(S^{k\sigma} \wedge H\underline{\mathbb{Z}})$

Note $\pi_i(S^{k\sigma} \wedge H\underline{\mathbb{Z}}) = \pi_{i+k}(S^{k\rho} \wedge H\underline{\mathbb{Z}})$ because $S^{k\rho} = \Sigma^k S^{k\sigma}$

We have a chain complex of $\mathbb{Z}[C_2]$-modules

γ = gen of C_2

$$\mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[C_2] \xleftarrow{1-\gamma} \mathbb{Z}[C_2] \xleftarrow{1+\gamma} \mathbb{Z}[C_2] \xleftarrow{1-\gamma} \mathbb{Z}[C_2] \leftarrow \cdots$$

$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4$

We stop at dim k. $H_*(C) = \tilde{H}_*(S^k) = \pi_*(S^k \wedge H\mathbb{Z})$

We need to take fixed points, i.e. apply the functor $\mathrm{Hom}_{\mathbb{Z}[C_2]}(\underset{a}{\mathbb{Z}}, -)$ trivial C_2-action

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow \cdots$$

$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4$

The H_* of this complex is $\pi_*^G(S^{kG} \wedge H\mathbb{Z})$

$$\pi_i\left(S^{k\sigma}\wedge H\underline{\mathbb{Z}}\right) = \begin{cases} \mathbb{Z} & \text{if } i = k = \text{even} \\ \mathbb{Z}/2 & \text{if } i \text{ is even and } 0 \le i < k \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\begin{aligned} &\pi_*^G(X) = \pi_*(X^G) \\ &\pi_*^G(S^{k\sigma}\wedge H\underline{\mathbb{Z}}) \\ &= \pi_*\left(\left(S^{k\sigma}\wedge H\underline{\mathbb{Z}}\right)^G\right) \end{aligned}}$$

These elements have names

For k even, the generator of π_k is $u_{k\sigma}$

$$S^2 \xrightarrow{u_{2\sigma}} S^{2\sigma}\wedge H\underline{\mathbb{Z}}$$

$$S^0 \xrightarrow{a_\sigma} S^\sigma$$

In lower even dimensions for k even we have $a_{2i\sigma}\, u_{(k-2i)\sigma} = a_\sigma^{2i}\, u_{2\sigma}^{(k/2-i)}$

From now on $u = u_{2\sigma}$ and $a = a_\sigma$

In $\pi_\bigstar MU$ we have $\bar{x}_1 \in \pi^G_{i\rho} MU$

which restricts to $x_i \in \pi_{2i} MU$

where $\pi_*(MU) = \mathbb{Z}[x_1, x_2 \ldots]$ $\quad x_i \in \pi_{2i}$

The E_2-term for our SS is

$$\mathbb{Z}[\bar{x}_1, \bar{x}_2, \ldots\ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[u, a] / (2a)$$

where $\quad \bar{x}_1 \in E_2^{0, i\rho}$ $\quad$ permanent cycle

$u \in E_2^{0, 2-2\sigma}$ $\quad$ not a cycle

$a \in E_2^{1, 1-\sigma}$ $\quad$ permanent cycle

On the 0-line with integer grading we have

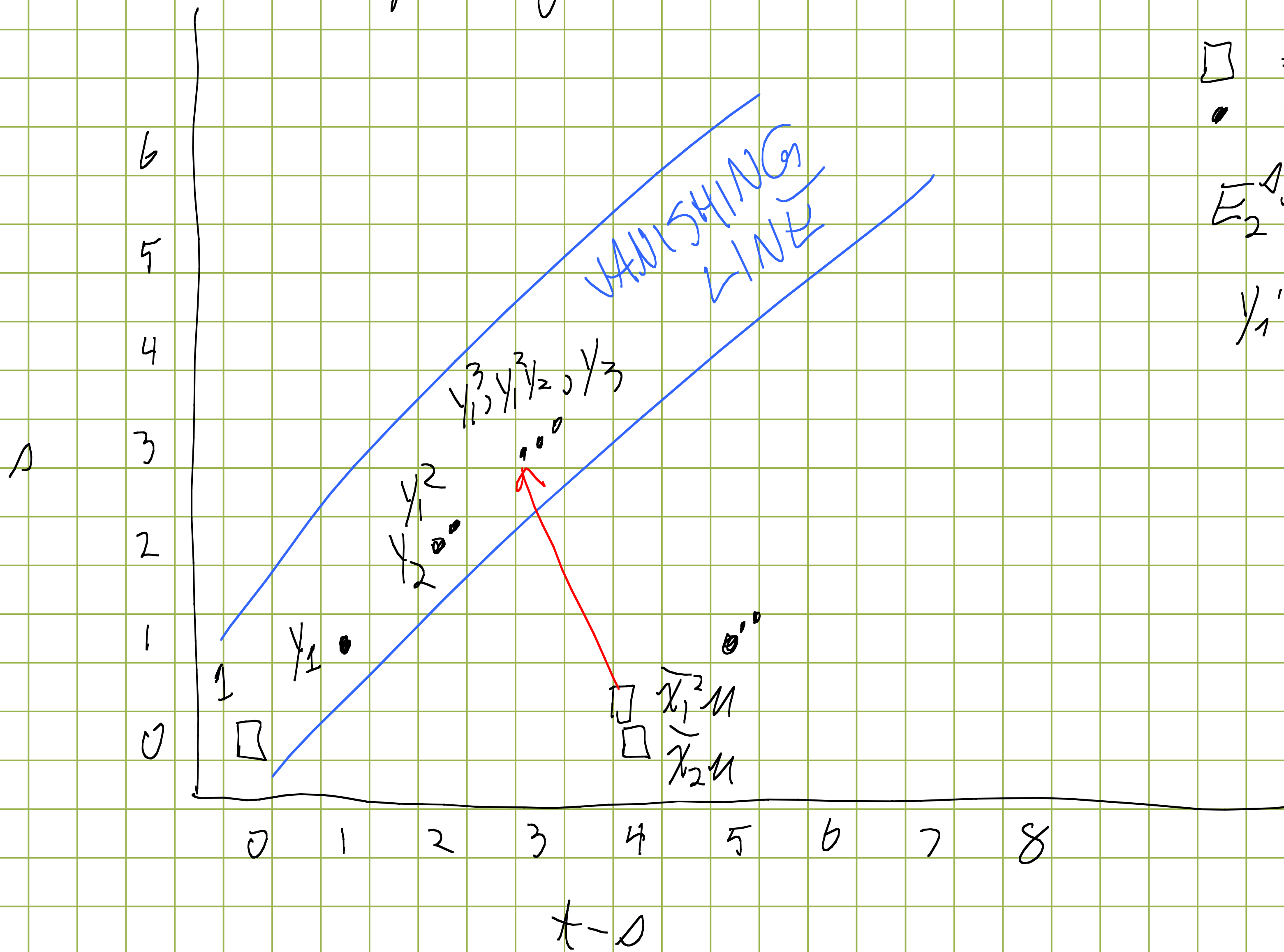
$$\mathbb{Z}[\bar{x}_1^2 u^i : i \text{ odd}] \otimes \mathbb{Z}[\bar{x}_{2i} u^i : i > 0]$$

$E_2^{0,4i}$

On the vanishing line of slope 1 we have

$$y_i = a^i \bar{x}_1 \in E_2^{i,\, i - i\sigma + i(1+\sigma)} = E_2^{i,2i}$$

Picture of $\mathbb{Z}$-graded in low dims



□ = $\mathbb{Z}$

• = $\mathbb{Z}/2$

$E_2^{s,t} =$

$y_1 \in E_2^{1,2}$

The vanishing line is $\mathbb{Z}[y_1, y_2, y_3, \dots] / (2y_i)$

The number of slices of the form $S^{k\rho} \wedge H\mathbb{Z}$
= the number of degree k-monomials in $\mathbb{Z}[x_1, x_2, \dots]$

<u>The tool for finding differentials</u>

Recall geometric fixed points

$$\Phi^G(MU) = MO$$

$$k \neq 2^j - 1$$

$$\pi_* MO = \mathbb{Z}/2[y_2, y_4, y_5 \cdots y_k \cdots]$$

The isotropy separation sequence

$$EC_{2+} \longrightarrow S^0 \longrightarrow \tilde{E}C_2 \qquad C_2 = G$$

$EC_2 =$ contractible free C_2-space
$= S^\infty$ with antipodal action

$X_+ = X \cup \{pt\}$

Thm $\pi_\star(X \wedge \tilde{E}C_2) = a^{-1}\pi_\star(X)$

and $\quad \Phi^{C_2} X = (\tilde{E}C_2 \wedge X)^{C_2}$

For $X = MU_{\mathbb{R}}$ then $\Phi^{C_2} MU = MO$.

It follows that if we invert $\bar{a}$ in the slice SS, it will converge to $\mathbb{Z}/2[a^{\pm}] \otimes \pi_*(MO)$

In particular y_1, y_3, y_7, etc must each be killed by some power of a.

$$y_i = a^i \bar{x}_1$$

The only pattern of differentials that does this is

$$\color{red}{2-2\sigma} \quad u \xrightarrow{d_3} a^3 \bar{x}_1 = a^2 y_1 \quad \color{red}{-3\sigma + 1 + \sigma = 1 - 2\sigma}$$

$$d_3(u^2) = 0 \qquad d_7(u^2) = a^7 \bar{x}_3 = a^4 y_3 \; \color{red}{\in \pi_{-4\sigma+3}}$$

$$d_{15}(u^4) = a^{15}\bar{x}_7 = a^8 y_7$$

etc

$$u \longrightarrow a^3\bar{x}_1 = a^2 y_1 \qquad\qquad y_1 = a\bar{x}_1$$

$$\bar{x}_1^2 u \longrightarrow a^2 y_1 \bar{x}_1^2 = y_1^3$$

What next?

What happens if we invert $\bar{x}_1$?

$$\bar{x}_1 : S^p \longrightarrow MU_{\mathbb{R}}$$

$$\begin{tikzcd} S^p \wedge MU_{\mathbb{R}} \arrow[r] \arrow[rr, bend right, "\bar{x}_1"'] & MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \arrow[r] & MU_{\mathbb{R}} \end{tikzcd}$$

$$MU_{\mathbb{R}} \xrightarrow{\bar{x}_1} S^{-p} MU_{\mathbb{R}} \xrightarrow{\bar{x}_1} S^{-2p} MU_{\mathbb{R}} \longrightarrow \cdots\cdots$$

We get a telescope $\bar{x}_1^{-1} MU_{\mathbb{R}}$

$$u \xrightarrow{d_3} a^3 \bar{x}_1$$

$$\bar{x}_1^{-1} u \xrightarrow{d_3} a^3$$

$$u^2 \xrightarrow{d_7} a^7 \bar{x}_3 = d_3 (\bar{x}_3 \bar{x}_1^{-1} a^4 u)$$

This means u^2 is now a perm cycle.