Note Title 3/15/2010

MATH 549 3-15-10

Main Theorem $\theta_j \in \pi^S_{2^{j+1}-2}$ does not exist for $j \geq 7$.

Strategy: Construct a spectrum Ω with

i) If θ_j exists, it has nontrivial image in $\pi_* \Omega$ DETECTION

ii) PERIODICITY $\pi_k \Omega = \pi_{k+256} \Omega$

iii) GAP $\pi_{-2} \Omega = 0$

The construction of Ω and the proofs of

ii) and iii) use equivariant methods.
The proof of i) uses algebra associated with formal A-modules.

Equivariant homotopy theory

A G-space X is a space with a G-action. A map $f: X \longrightarrow Y$ between G-spaces is <u>equivariant</u> if it respects the G-actions on X and Y.

<u>Example</u> Let X be a complex algebraic variety that is defined over $\mathbb{R}$,

such $\mathbb{C}P^n$. We get an action of C_2 via complex conjugation, i.e.

$$[z_0, z_1, \dots z_n] \longmapsto [\bar{z}_0, \bar{z}_1, \dots \bar{z}_n].$$

Associated with a G-space one has

* fixed point sets for $H \subset G$

  $X^H = \{ x \in X : h(x) = x \quad \forall\, h \in H \}$ = subspace of X

* isotropy groups for $x \in X$

  $G_x = \{ g \in G : g(x) = x \}$ = subgp of G

* orbits $Gx = \{ g(x) : g \in G \}$

  Each orbit has the form G/G_x

(the set of left/right cosets of G_x

For technical reasons, one often wants a base point fixed by G. For a G-space X, $X_+ = X \cup \{\text{base point}\}$ with base pt fixed by G, e.g. G_+

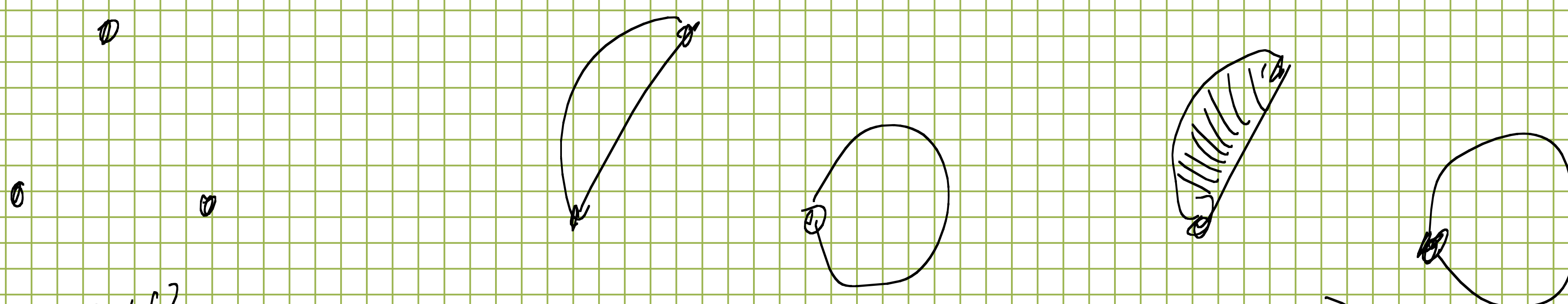
<u>Def</u> Cin equiv map: $X \xrightarrow{f} Y$ is a weak G-equivalence if $X^H \xrightarrow{f^H} Y^H$ is a weak equivalence for each subgp $H < G$.

Two ways to construct G-CW cxs.

Recall a CW cx is a space X constructed as follows

X^0 = discrete,

X^1 is obtained from X^0 by adjoining paths between points in X^0



X^0 $\qquad$ X^1 $\qquad$ X^2

X^n is obtained from X^{n-1} by attaching n-cells (copies of D^n) to X^{n-1} via maps

$$\partial D^n \longrightarrow X^{n-1}$$

Method 1. Use orbits. Replace cells D^n by either $G/H \times D^n$ (unbased case) or $G/H_+ \wedge D^n$ (based case)

Replace the pair (D^n, S^{n-1}) by

$$(G/H \times D^n, \; G/H \times S^{n-1}) \quad \text{or}$$

$$(G/H_+ \wedge D, \; G/H_+ \wedge S^{n-1})$$

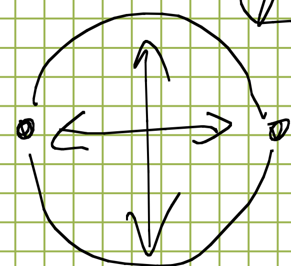
One can build a G-CW cx from

these using equivariant attaching maps. G acts by permuting cells.

<u>Example</u> $G = C_2$, $X = S^n$ with antipodal action, no base point.

$X^0 = C_2$

$$\boxed{\text{free } i\text{-cell} = C_2 \times D^i}$$

$X^1 =$ 

X^i is obtained from X^{i-1} by adjoining $C_2 \times D^i$

This complex has one free C_2-cell in each dimension.

Method 2. Use representations.

V = real orthogonal representation of G

= finite dim real vector space with a linear action of G.

$$D(V) = \{x \in V : \|x\| \le 1\}$$

$$S(V) = \{x \in V : \|x\| = 1\} = \partial D(V)$$

unbased G-spaces

$(D(V), S(V))$ = cell pair.

One can build CW-cxs out of these.

Each cell is invariant.

Notation $S^V = D(V)/S(V)$

$=$ one pt compactification of V

Method 3

Let V be a rep of a subgp $H \subset G$

$(G \times_H D(V),\ G \times_H S(V))$ $(hg, y) \sim (g, h^{-1}y)$

What is $G \times_H Y$ for an H-space Y?

$h(g, x) = (gh^{-1}, hx)$

so it acts freely on $G \times Y$

$G \times_H Y = (G \times Y)/H$. G acts on thus by left multiplication.

Toward G-spectra

A **prespectrum** E was a collection of spaces E_n and maps $\Sigma E_n \to E_{n+1}$. It is a spectrum if each adjoint map $E_n \to \Omega E_{n+1}$ is a homeomorphism.

One converts a prespectrum $\{E_n\}$ to a spectrum $\{\tilde{E}_n\}$ by defining

$$\tilde{E}_n = \varinjlim_k \Omega^k E_{n+k}$$

It is convenient to have spaces indexed by finite dimensional subspaces V of fixed infinite dimensional vector U called a universe. The htpy type of $E(V)$ depends only on $\dim V$.

One has structure maps

$$S^W \wedge E(V) = \Sigma^{|W|} E(V) \longrightarrow E(V+W)$$

for orthogonal subspaces V and W.

This is an orthogonal spectrum

Technical point: In order to a spectrum a prespectrum, it is enough to define $E(W)$ for a cofinal collection of fin dim vector spaces $W \subset U$, i.e. a collection $\{W\}$ is one for which each V is contained in some W.

We spaces $E(V)$ and maps $S^W \wedge E(V) \to E(V+W)$ for $W \perp V$.

Going equivariant:

* Each V is a rep. of G.

* The universe U is <u>complete</u> if it contains every finite dim rep of G as a subspace
* $E(V)$ is a based G-space, as is S^W and each structure map $S^W \wedge E(V) \to E(V+W)$

Minor technical point
Suffices to define $E(V)$ for a cofinal collection of reps V.

Example MU as a C_2-spectrum
$U = \mathbb{C}^\infty$ with conjugation

Every fin dim rep of C_2 is a direct sum $V = V_+ \oplus V_-$ where C_2 acts trivial on V_+ and by the sign rep on V_-. Let $\rho = \mathbb{C} = \mathbb{R}_+ \oplus \mathbb{R}_-$

ρ = regular rep of C_2

σ = 1-dim sign rep

$\rho = 1 + \sigma$ in $RO(C_2)$ = real rep. ring of C_2

The collection $\{\mathbb{C}^n\} = \{n\rho\}$ is cofinal

$MU(\mathbb{C}^n) = MU(n)$ with conjugation