MATH 549 3-1-10

Note Title 3/1/2010

Recall the spectra MU and MO

Cultural excursion

$BU(n)$ and $BO(n)$ are classifying spaces for cx + real n-plane bundles.

Thm Let X be a paracompact space. Then the set of iso classes of $\mathbb{C}^n$ bundles over X is bijective with the set of homotopy classes of maps $X \xrightarrow{f} BU(n)$.

Recall $BU(n)$ has a canonical $\mathbb{C}^n$-bundle over it. $BU(n)$ = space of n-planes in $\mathbb{C}^\infty$. Each has a $\mathbb{C}^n$ associated with it

REFERENCE: Milnor + Stasheff — Characteristic Classes

We a map

$$\begin{array}{c} \mathbb{C}^\infty \times BU(n) \\ \cup \\ E \xrightarrow{p} BU(n) \end{array}$$

with

$p^{-1}(x)$ = all vectors in the n-plane x.

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BU(n) \end{array}$$

$f^*E = \{(x, e) \in E \times X : f(x) = p(e)\}$

This is a $\mathbb{C}^n$ bundle over X.

Let M^n be a closed C^∞-manifold embedded in some $\mathbb{R}^{n+2k}$ so that its normal bundle ν has a complex structure.

Then ν is induced by a map $M \xrightarrow{f} BU(k)$.

M has a tubular neighborhood U in $\mathbb{R}^{n+2k}$ with a map $p: U \to M$ which is a $\mathbb{C}^k$-bundle. Hence we have diagram

$$\begin{array}{ccc} U & \longrightarrow & E \\ p\downarrow & & \downarrow p \\ M & \xrightarrow{f} & BU(k) \end{array}$$

The induced bundle over M is the normal bundle ν.

This leads to a map from U', the one point compactification of U to $MU(k)$, the one point compfn of E.

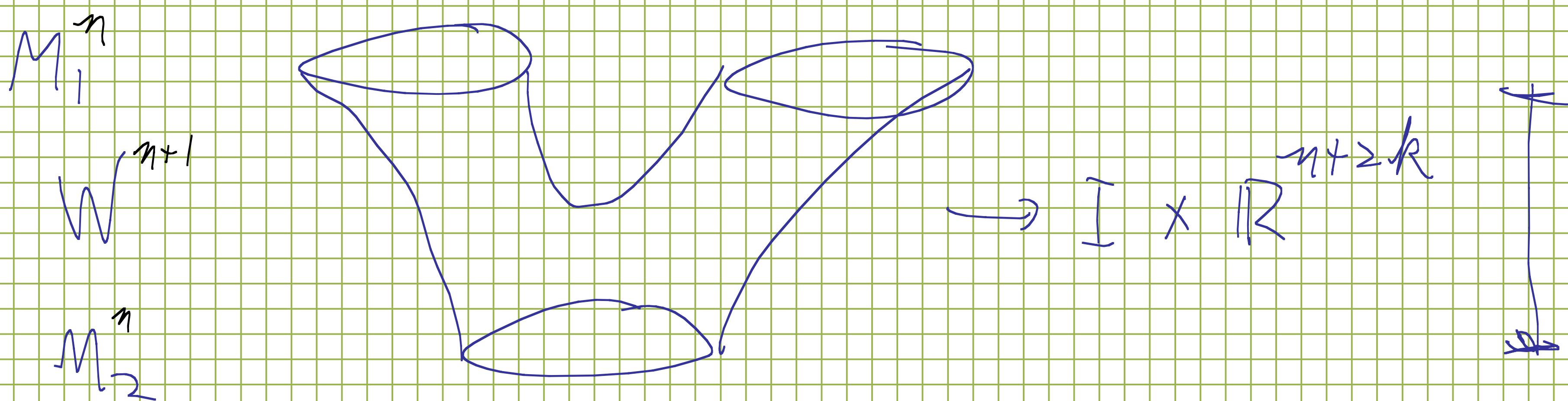
We also have a map

$$\begin{tikzcd}
S^{n+2k} \arrow[r] \arrow[d, equal] & U' \arrow[r] & MU(k) \\
\mathbb{R}^{n+2k} \cup \{\infty\} & & \\
U \arrow[u, hook] \arrow[r] & U \arrow[uu] &
\end{tikzcd}$$

We get an element in

$$\pi_{n+2k} MU(k) \longrightarrow \pi_n \underline{MU} = \varinjlim_{k} \pi_{n+2k} MU(k)$$

Def Two such manifolds M_1^n and M_2^n are complex cobordant if there is a mfd W^{n+1} with $\partial W = M_1^n \cup M_2^n$



such W has a complex normal bundle in $I \times \mathbb{R}^{n+2k}$ which restricts to the normal bundles of M_1 and M_2

Theorem (Thom) M_1 and M_2 are cobordant iff the two maps $S^{n+2k} \longrightarrow MU(k)$ are homotopic. The group (under disjoint union) of cobordism classes of such manifolds is $\pi_{n+2k} MU(k)$.

Each element of $\pi_* MU$ corresponds to a cobordism class of "complex" mfds.

Each element of $\pi_* MO$ corresponds to an unoriented cobordism class.

$MU_*(X)$ for a space X is the graded gp of cobordism classes of maps $M \longrightarrow X$.

$$\begin{tikzcd}
M_1^n \arrow[dr] \arrow[drr, bend left, "f_1"] & & \\
& W^{n+1} \arrow[r, "g"] & X \\
M_2^n \arrow[ur] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

f_1 and f_2 are bordant if

Back to the main story

$MU^*(\mathbb{CP}^\infty) = MU^*(pt)[[x]] \qquad x \in MU^2(\mathbb{CP}^\infty)$

$MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = MU^*(pt)[[x\otimes 1, 1\otimes x]]$

$\mathbb{CP}^\infty = BU(1) = K(\mathbb{Z},2)$ is a topological group, so there is a map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$

This map induces the tensor product of complex line bundles. This map induces

$$MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \longleftarrow MU^*(\mathbb{C}P^\infty) \qquad R = MU^*(pt)$$

$$R[[x\otimes 1, 1\otimes x]] \qquad\qquad R[[x]]$$

$$F(x\otimes 1, 1\otimes x) := \sum a_{ij}\, x^i \otimes x^j \longleftarrow\!\!| \; x$$

$$a_{ij} \in MU^{2(1-i-j)}(pt) = \pi_{2(i+j-1)}(MU).$$

Properties of the power series

$$F(x,y) \in R[[x,y]]$$

1) $F(x,0) = F(0,x) = x$

2) $F(x,y) = F(y,x)$

3) $F(x, F(y,z)) = F(F(x,y), z)$ associativity

Such an F is called a <u>formal group law</u> over R.

<u>Examples</u> 1) $F(x,y) = x+y$ Additive FGL

2) $F(x,y) = x+y+xy$ Multiplicative FGL

$$1+F(x,y) = (1+x)(1+y)$$

3) $F(x,y) = \dfrac{x+y}{1+xy} = (x+y)\sum_{n=0}^{\infty} (-1)^n x^n y^n$

$\tan(\alpha+\beta) = F(\tan(\alpha), \tan(\beta))$

Lazard's universal example

suppose $F(x,y) = \sum a_{ij} x^i y^j$ for $a_{ij} \in R$

subject to the 3 conditions

Let $L = \mathbb{Z}[\hat{a}_{ij}]/(\text{relations})$. This ring has a universal FGL over it given above. For any FGL over any R, there is a unique hom $\theta : L \longrightarrow R$.

$\hat{a}_{ij} \longrightarrow a_{ij}$

Quillen's Theorem (1969): For the FGL over $MU^*(pt)$, θ is an isomorphism.

$MU_* X = \pi_*(MU \wedge X)$ = complex bordism group of X

$MU^* X = [X, MU]$ = complex cobordism of X