MATH 549 2-22-10

Recall a mod 2 Adams resolution is a diagram

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\ L_0 & & L_1 & & L_2 & & \end{array}$$

$H/2$ = mod 2 Eilenberg-Mac Lane spectrum

where

1) Each L_s is a wedge of $\Sigma^? H/2$ so $H^*(L_s)$ is a free A-module
2) $H^*(f_s)$ is onto

3) X_{s+1} is the fiber of f_s (so $H^*(g_s) = 0$).

This leads to a LES

$$0 \leftarrow H^* X \leftarrow H^* L_0 \leftarrow H^* \Sigma L_1 \leftarrow H^* \Sigma^2 L_2 \leftarrow \cdots$$

a free A-resolution of $H^* X$

Lemma A map $S^k \xrightarrow{x} X = X_0$ lifts to X_s iff it has Adams filtration $\geq s$, i.e. it is the composite of s maps each inducing 0 in $H^*(\,;\mathbb{Z}/2)$.

$$\begin{tikzcd}
 & S^k \arrow[dl, dashed, "x_0"'] \arrow[d, dashed, "x_1"] \arrow[dr, dashed, "x_2"] & & \\
X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \cdots \arrow[l, "g_2"'] \\
L_0 & L_1 & L_2 & \arrow[l] \text{known homotopy}
\end{tikzcd}$$

Want to find $\pi_* X$

Pf. One direction is clear since $H^*(g_0) = 0$

For the converse, suppose $H^*(x_0) = 0$

Then $f_0 x_0$ is trivial, so $\exists\, x_1$ since X_1 is the fiber of f_0. This is the converse for $s = 1$. Similar arguments work for all s. QED.

Let $L_{s,m}$ be the cofiber of the composite map $X_{s+m} \to X_s$, so $L_{s,1} = L_s$

$$\begin{tikzcd}
X_{s+m+1} \arrow[r] \arrow[d, "g_{s+m}"'] & X_s \arrow[r] \arrow[d, "="] & L_{s,m+1} \arrow[d] & \\
X_{s+m} \arrow[r] \arrow[d, "f_{s+m}"'] & X_s \arrow[r] \arrow[d] & L_{s,m} \arrow[d] & S^k \arrow[l] \arrow[ul, dashed] \arrow[dl, "\text{obstruction}"] \\
L_{s+m} \arrow[r] & \text{point} \arrow[r] & \Sigma L_{s+m} &
\end{tikzcd}$$

Each row + column is a fiber sequence

<u>Question</u>: Given an elt $S^k \xrightarrow{Y} L_s = L_{s,1}$ can we lift it to $L_{s,2}$, $L_{s,3}$ ....

For each n there is a possible obstruction. The spectral sequence keeps track of these.

What happens: Our initial diagram leads to a cochain complex

$$\begin{array}{ccccccc} E_1^{0,t} & & E_1^{1,t} & & E_1^{2,t} & & \\ \| & & \| & & \| & & \\ \pi_t L_0 & \xrightarrow{d_1^{0,t}} & \pi_t \Sigma L_1 & \xrightarrow{d_1^{1,t}} & \pi_t \Sigma^2 L_2 & \xrightarrow{d_1^{2,t}} & \cdots \end{array}$$

for each t

$E_1^{s,t} = \pi_t \Sigma^s L_s = \pi_{t-s} L_s$ = "known"

We cohomology gps

$$E_2^{s,t} = \ker d_1^{s,t} / \operatorname{im} d_1^{s-1,t}$$

= better known.

These gps are independent of the choices made, i.e. they depend only on X.

Chasing diagrams shows we have another cochain α (for each t)

$$\cdots \longrightarrow E_2^{s-2,t-1} \xrightarrow{d_2^{s-2,t+1}} E_2^{s,t} \xrightarrow{d_2^{s,t}} E_2^{s+2,t+1} \longrightarrow \cdots$$

We define its cohomology to be

$$E_3^{s,t} = \ker d_2^{s,t} / \operatorname{im} d_2^{s-2,t-1}$$

$$\vdots$$

$$\cdots \longrightarrow E_r^{s-r,t-r-1} \xrightarrow{d_r^{s-r,t-r-1}} E_r^{s,t} \xrightarrow{d_r^{s,t}} E_r^{s+r,t+r-1} \longrightarrow \cdots$$

d_r raises s by r

and lowers $t-s$ by 1

In practice one can show for a given (s,t) that d_r (both coming and going) vanishes for large r, so $E_r^{s,t} = E_\infty^{s,t}$. This gp is a subquotient of $\pi_{t-s}(X)$, namely $F^s\pi_{t-s} / F^{s+1}\pi_{t-s}$ where

$F^s\pi_*(X)$ = all maps which can be factored as a composite of s maps that trivial in H^*.

How to compute for $X = S^0$.

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X), \mathbb{Z}/2) = \mathrm{Ext}_A(\mathbb{Z}/2, \mathbb{Z}/2)$$

We find this using a free A-resolution of $\mathbb{Z}/2$

$$\begin{array}{ccccccc} & & & 0 & & 1 & \\ 0 \leftarrow & \mathbb{Z}/2 & \xleftarrow{\varepsilon} & A & \longleftarrow & A\{\bar{h}_j : j \geq 0\} & \longleftarrow \cdots \\ & & & Sq^{2^j} & \longleftarrow\!\shortmid & \bar{h}_j & \end{array}$$

$\ker \varepsilon$ is generated as an A-module by $\{Sq^{2^j} : j \geq 0\}$.

Hence we get $h_j \in \mathrm{Ext}_A^{1,2^j}(\mathbb{Z}/2, \mathbb{Z}/2)$

$=: \mathrm{Ext}^{1,2^j} = E_2^{1,2^j}$

These form a basis of $\mathrm{Ext}^{1,*} = E_2^{1,*}$, the 1-line.

$R \wedge R \to R$

$S^0 \wedge S^0 \to S^0$

Multiplicative structure

Because S^0 is a ring spectrum, each E_n^{**} is an algebra, i.e. $\exists$ maps

$$E_n^{s',t'} \otimes E_n^{s'',t''} \longrightarrow E_n^{s'+s'',t'+t''}$$

with nice properties.

We have $h_j \in E_2^{1,2^j}$

$$h_j h_k \in E_2^{2,2^j+2^k}$$

The 2-line $(E_2^{2,*})$ has basis

$$\{h_j h_k : 0 \leq j \leq k \text{ and } k \neq j+1\}$$

$$h_j h_{j+1} = 0$$

Browder's Thm ^{1969} There is framed mfd of Kervaire invariant 1 in dim $2^{j+1}-2 \iff h_j^2$ is a permanent cycle in the Adams SS, i.e it lives to E_∞, i.e. it does not support a nontrivial differential.

Thm (Adams 1961)

$$d_2(h_j) = h_0 h_{j-1}^2 \neq 0 \text{ for } j \geq 4$$

Only h_0, h_1, h_2 and h_3 survive.