MATH 549 2-17-10

Note Title 2/17/2010

Recall we want compute $\pi_{n+k} X$ where X is $(n-1)$-connected and $k < n$.

How to let $n \to \infty$: the category of spectra

<u>Def</u> A <u>prespectrum</u> <u>X</u> is a collection of spaces X_n and maps $\Sigma X_n \xrightarrow{i_n} X_{n+1}$ for $n \geq 0$.

$X_n \xrightarrow{\hat{i}_n} \Omega X_{n+1}$

<u>Examples:</u> ① $X_n = S^n$ and i_n is an equivalence

<u>H/2</u> ② $X_n = K_n = K(\mathbb{Z}/2, n)$ i_n is not an equivalence, but $\hat{i}_n$ is.

We define $H_k(\underline{X}) = \lim_{n\to\infty} H_{n+k}(X_n)$

i_n induces a hom $H_{n+k}(X_n) \longrightarrow H_{n+1+k}(X_{n+1})$

$$\pi_k(\underline{X}) = \lim_{n\to\infty} \pi_{n+k}(X_n)$$

1. $H_k(\underline{S}^0) = \begin{cases} \mathbb{Z}/2 & \text{for } k=0 \\ 0 & \text{for } k\neq 0 \end{cases}$

2. $\pi_k(\underline{H/2}) = \begin{cases} \mathbb{Z}/2 & \text{for } k=0 \\ 0 & \text{for } k\neq 0 \end{cases}$

$\pi_k(\underline{S}^0) =$ mystery

$H^*(\underline{H/2}) = A$ as a graded vector space

Pre def: A map $\underline{X} \xrightarrow{f} \underline{Y}$ is a collection

$$\begin{tikzcd}
\Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d] & \Sigma Y_n \arrow[d] \\
X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1}
\end{tikzcd}$$

for $n \geq 0$.

(Too restrictive)

How to get from a prespectrum $\underline{\tilde{X}}$ to a spectrum $\underline{X}$:

We have $\tilde{X}_n \xrightarrow{\hat{\imath}_n} \Omega \tilde{X}_{n+1}$ for $n \geq 0$

$$\tilde{X}_n \xrightarrow{\hat{\imath}_n} \Omega \tilde{X}_{n+1} \xrightarrow{\Omega \hat{\imath}_{n+1}} \Omega^2 \tilde{X}_{n+2} \xrightarrow{\Omega^2 \hat{\imath}_{n+2}} \Omega^3 \tilde{X}_{n+3} \longrightarrow \cdots$$

Let $X_n = \lim_{i\to\infty} \Omega^i \tilde{X}_{n+i}$. Then X_n is homeomorphic to ΩX_{n+1}. We can define $X_{n-k} = \Omega^k X_n$, so X_n is defined for all $n \in \mathbb{Z}$.

We can define $H_*(\underline{X})$, $H^*(\underline{X})$ and $\pi_*(\underline{X})$ as before, and the predef of a map is the right one.

<u>Def</u> A spectrum $\underline{X}$ is <u>connective</u> if $\pi_i(\underline{X}) = 0$ for $i < 0$.

Example of a nonconnective spectrum

$U(n) = n$th unitary gp

$\quad = $ gp of $n\times n$ unitary matrices$/\mathbb{C}$.

$U(n) \hookrightarrow U(n+1)$

$U = \lim\limits_{n\to\infty} U(n) = $ stable unitary gp

Bott Periodicity Thm

$\Omega^2 U \simeq U$ and $\pi_i U = \begin{cases} \mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$

$\Omega^8 SO \simeq SO$ and $\pi_i SO = \begin{cases} \mathbb{Z} & \text{if } i \equiv 3 \text{ or } 7 \bmod 8 \\ \mathbb{Z}/2 & \text{if } i \equiv 0, 1 \quad " \\ 0 & \text{otherwise} \end{cases}$

The unitary gp spectrum $\underline{X} = \underline{U}$

$$X_n = \begin{cases} U & \text{if } n \text{ is even} \\ \Omega U & \text{if } n \text{ is odd} \end{cases}$$

$$\begin{array}{ccccccccc}
X_0 & \longrightarrow & \Omega X_1 & \longrightarrow & \Omega^2 X_2 & \longrightarrow & \Omega^3 X_3 & \longrightarrow & \cdots \\
\| & & \| & & \| & & \| & & \\
U & \xrightarrow{\simeq} & \Omega^2 U & \xrightarrow{\simeq} & \Omega^2 U & \xrightarrow{\simeq} & \Omega^4 U & \longrightarrow & \cdots
\end{array}$$

$$\pi_i \underline{U} = \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{array} \right\} \text{ for all } i \in \mathbb{Z}$$

Properties of spectra

We can desuspend a spectrum $\underline{X}$

$\underline{Y} = \Sigma^{-1}\underline{X}$ is defined by $Y_n = X_{n-1}$

$\quad\;\; \shortparallel$

$\quad\; \Omega\underline{X}$

$\underline{W} = \Sigma X$ " $W_n = X_{n+1}$

Fiber sequences and cofiber sequences are the same thing.

$\pi_n(\underline{X}) = \pi_n(X_0)$
for $n \geq 0$

$\pi_k(\underline{X}) = \pi_{n+k}(X_n)$

Back to the Adam resolution

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \arrow[l] \cdots \\ L_0 & L_1 & L_2 & \end{tikzcd}$$

1) L_s is a wedge of suspensions of $H/2$

This means $H^*(L_s; \mathbb{Z}/2)$ is a free A-module and $\pi_*(L_s) = \mathrm{Hom}_A(H^*(L_s), \mathbb{Z}/2)$ as graded gps

2) $H^*(f_s)$ is onto

3) X_{s+1} is the fiber of f_s

This leads to a LES of A-modules

$$0 \leftarrow H^* X \leftarrow H^* L_0 \leftarrow H^* \Sigma^{-1} L_1 \leftarrow H^* \Sigma^{-2} L_2 \leftarrow \cdots$$

This is a free (hence projective) A-resolution of $H^* X$.

Recollections from homological algebra.

Let M and N be R-modules. M has a projective (or free) resolution

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

This leads to a cochain complex

$$\mathrm{Hom}_R(P_0, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow \mathrm{Hom}_R(P_2, N) \longrightarrow \cdots$$

Its cohomology depends only on M and N, not on the choice of P_i.

$$H^s =: \mathrm{Ext}^s_R(M, N)$$

$s \geq 0$

If R, M and N are all graded, then so is this group.

In our case:

$R = A$ (graded)

$M = H^* X$ $\quad N = \mathbb{Z}/2$

$P_s = H^*(\Sigma^{-s} L_s)$

$$\mathrm{Hom}_R(P_s, N) = \mathrm{Hom}_A(H^*\Sigma^{-s}L_s, \mathbb{Z}/2) = \pi_*(\Sigma^{-s}L_s)$$

$$\begin{tikzcd}
X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \cdots \arrow[l] \\
L_0 & L_1 & L_2 &
\end{tikzcd}$$

Adams resolution

Technical Lemma. Suppose X is connective and of finite type (each $\pi_k X$ is finitely generated). Let $\hat{X}$ be the cofiber of $\varprojlim X_s \longrightarrow X \longrightarrow \hat{X}$. Then $\hat{X}$ is the 2-adic completion of X, i.e.

$$\pi_* \hat{X} = \pi_* X \otimes \mathbb{Z}_2.$$

where $\mathbb{Z}_2$ = 2-adic integers.

<u>Theorem</u> (Adams 1959) There is a spectral sequence converging to $\pi_* \hat{X}$ with

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}\left(H^*(X), \mathbb{Z}/2\right)$$

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,\, t+r-1}$$

$E_\infty^{s,t}$ is a subquotient of $\pi_{t-s}(\hat{X})$.

The (decreasing Adams filtration on $\pi_* \hat{X}$ is defined as follows: An element $x \in \pi_k(X)$ has filtration $\geq s$ if it is rep'd by a map which can be factored

$$\begin{tikzcd}
S^k \arrow[rrrr, "x"] \arrow[d, "x_s"'] & & & & X \\
Y_{s-1} \arrow[r, "x_{s-1}"] & Y_{s-2} \arrow[r, phantom, "\cdots"] & \arrow[r] & Y_1 \arrow[ur, "x_1"'] &
\end{tikzcd}$$

with $H^*(x_i) = 0$

Example ① $X = S^0$ and $x =$ identity map $S^0 \to S^0$

Since $H^*(\iota) \neq 0$, the above does not occur for any $s > 0$.

② $x = 2^s \iota$

$$\underbrace{\overbrace{S^0 \xrightarrow{2} S^0 \xrightarrow{2} S^0 \xrightarrow{2} S^0 \cdots \xrightarrow{2} S^0}^{x}}_{s \text{ factors}}$$

This map has filtration s.