MATH 549

Note Title 2/15/2010

Browder's thm says ∃ a mfd with Kervaire invariant one in dim $2^{r+1}-2$ iff ~~~~~~~~.

Recall Steenrod operations

$$Sq^k : H^n(X;\mathbb{Z}/2) \longrightarrow H^{n+k}(X;\mathbb{Z}/2)$$

$$Sq^k x = \begin{cases} 0 & \text{if } n < k \\ x^2 & \text{if } n = k \\ ? & \text{if } n > k \end{cases}$$

These generate the mod 2 Steenrod algebra A.

Let $Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_k}$

It is <u>admissible</u> if $i_1 \geq 2i_2$, $i_2 \geq 2i_3$, etc.

The admissible monomials form a basis of A.

Def. The <u>excess</u> $e(I)$ of this monomial is

$$(i_1 - 2i_2) + (i_2 - 2i_3) + \cdots (i_{k-1} - 2i_k) + i_k$$

$$= i_1 - i_2 - i_3 \cdots - i_k$$

Recall $K_m = K(\mathbb{Z}/2, m)$

and let $H^*(X) = H^*(X; \mathbb{Z}/2)$

Thm $H^* K_m = \mathbb{Z}/2[Sq^I x_m : e(I) < m]$

and $x_m \in H^m K_m$ is the fundamental class, i.e. it corresponds to

$$H^m(K_m) = [K_m, K_m] \ni \text{identity}.$$

Example $m = 1$. The only admissible monomial with excess 0 is 1, so we have $H^*(K_1) = \mathbb{Z}/2[x_1]$ $\quad x_1 \in H^1$

$K_1 = \mathbb{RP}^\infty$

$n = 2$. The monomials with excess 1 are

Sq^1, $Sq^2 Sq^1$, $Sq^4 Sq^2 Sq^1$, etc, in dimensions

1, 3, 7, 15, 31, ...

We get $H^* K_2 = \mathbb{Z}/2[y_2, y_3, y_5, y_9, y_{17} \cdots]$

$y_2 = x_2$

$y_2 \xrightarrow{Sq^1} y_3 \xrightarrow{Sq^2} y_5 \xrightarrow{Sq^4} y_9 \xrightarrow{Sq^8} y_{17}$

where $y_n \in H^n$

Serre's method of computing $\pi_* X$.

Let X be $(n-1)$-connected for $n > 1$.

Hurewicz thm says $\pi_n(X) = H_n(X; \mathbb{Z})$

Let $L_0 = K(\pi_n(X), n)$. This space

has exactly one nontrivial homotopy grp, $\pi_n(L_0) = \pi_n(X)$. There is a map $X_0 = X \xrightarrow{f_0} L_0$ inducing this isomorphism.

Let X_1 be the homotopy theoretic fiber of f_0. We have a fiber sequence $\quad X_1 \xrightarrow{g_0} X_0 \xrightarrow{f_0} L_0$

There is a long exact sequence in π_*

$\cdots \to \pi_i(X_1) \xrightarrow{g_{0*}} \pi_i(X_0) \xrightarrow{f_{0*}}$ [illegible] $\to \pi_{i-1}(X_1) \to \cdots$

f_{0*} is an iso in dim n and trivial in other dimensions

It follows that

$$\pi_i(X_1) = \begin{cases} 0 & \text{for } i \le n \\ \pi_i(X) & \text{for } i > n \end{cases}$$

Suppose we know $H_*(X_1)$. The first nontrivial gp occurs above dim n, say n'.

Let $L_1 = K(\pi_{n'}(X), n')$. There is map $f_1 : X_1 \longrightarrow L_1$ inducing an iso in $\pi_{n'}$, so we get a fiber X_2 with $\pi_i(X_2) \cong \pi_i(X)$ for $i > n'$.

We could iterate this procedure and get a diagram

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_1"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_2"'] \arrow[d, "f_2"'] & X_3 \arrow[l, "g_3"'] \arrow[d, "f_3"'] & \cdots \arrow[l, "g_3"'] \\ L_0 & L_1 & L_2 & L_3 & \end{tikzcd}$$

where

1. Each L_i is a $K(\pi, n)$ for some π and n
2. f_i induces an iso in the first nontrivial homotopy group
3. X_{i+1} is the fibers of f_i.

We need to H_* of all spaces in right.

How to compute these homology groups

Suppose we have a fiber sequence

$$F \xrightarrow{g} E \xrightarrow{f} B$$

F = fiber of f. The Serre spectral sequence computes $H_*(E)$ in terms of $H_*(F)$ and $H_*(B)$. It can be used to find H_*F or H_*B if the other two are known.

Special case: Suppose F and B are both n-connected. Then below dimension $2n$, there is a LES

$$\cdots \to H_i(F) \to H_i(E) \to H_i(B) \to H_{i-1}(F) \to \cdots$$

Adams method (1959)

Assume all space are very highly connected, so their homology gps are related by LESs as above.

Recall $H^*(X;\mathbb{Z}_2)$ is a module over A.

Choose a set $x_1, x_2, x_3 \cdots \in H^*(X)$ that generate it (in our range of dimensions) as an A-module. For each such $x_i \in H^{m_i}(X)$ we get a map $X \to K_{m_i}$

$K_m = K(\mathbb{Z}/2, m)$. Let $L_0 = \prod_{i=1}^{?} K_{m_i}$

and $f_0 : X_0 \xrightarrow{\overset{X}{\shortparallel}} L_0$. This means that

$f_0^* : H^* L_0 \longrightarrow H^* X$ is onto

As before, let X_1 be the fiber of f_0. Our long exact sequence reduces to

$$0 \leftarrow H^i X_0 \leftarrow H^i L_0 \leftarrow H^{i-1} X_1 \leftarrow 0$$

$$0 \longleftarrow H^*X_0 \xleftarrow{h_0^*} H^*L_0 \xleftarrow{g_0^*} H^{*-1}X_1 \longleftarrow 0$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$

free A-module on the generators $x_1, x_2, \ldots\ldots$

We can iterate this construction

We get a diagram

$$
\begin{tikzcd}
X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & X_3 \arrow[l] \arrow[d, "f_3"'] & \cdots \arrow[l] \\
L_0 & L_1 & L_2 & L_3 &
\end{tikzcd}
$$

Adams resolution for X

where

1. $H^* L_i$ is a free A-module
2. f_i^* is onto
3. X_{i+1} is the fiber of f_i.

We get SES

$$0 \leftarrow H^* X_0 \leftarrow H^* L_0 \leftarrow H^{*-1} X_1 \leftarrow 0$$

$$0 \leftarrow H^{*-1} X_1 \leftarrow H^{*-1} L_1 \leftarrow H^{*-2} X_2 \leftarrow 0$$

etc

These can be spliced into a LES

$$0 \leftarrow H^*X \leftarrow H^*L_0 \leftarrow H^{*-1}L_1 \leftarrow H^{*-2}L_2 \leftarrow \cdots \quad (1)$$

free A-modules

This is a free A-resolution of H^*X.

and $\pi_t L_i = \mathrm{Hom}_A(H^*(L_i), \Sigma^t \mathbb{Z}/2)$

note $\mathrm{Hom}_A(A, \mathbb{Z}/2) = \mathbb{Z}/2$

The functor $\mathrm{Hom}_A(-;\mathbb{Z}/2)$ identifies the generators of the module

apply this functor to (1) and get

a cochain complex

$$\pi_*(L_0) \longrightarrow \pi_{*-1}(L_1) \longrightarrow \pi_{*-2}(L_2) \longrightarrow \cdots$$

<u>Theorem</u> The cohomology of this complex depends only on X, not on the choices of the L_i's.

This cohomology is

$$\mathrm{Ext}_A(H^*(X), \mathbb{Z}/2)$$

WHAT IS IT ???