MATH 549 1-25-10

Note Title 1/25/2010

Pontryagin-Thom construction

Given a framed closed smooth k-mfd $M^k \subset \mathbb{R}^{n+k}$

(framed means M^k has a closed nbd $W \cong M^k \times D^n$ and we have a homeomorphism)

$$\begin{tikzcd}
W \arrow[rr, "\text{project}"] \arrow[d, hook] & & D^n \arrow[d] \\
\mathbb{R}^{n+k} \arrow[r] & S^{n+k} \arrow[r, "g"] & S^n \\
S^{n+k} - W \arrow[rr] & & \infty \arrow[u]
\end{tikzcd}$$

Hence we get

$$g : S^{n+k} \to S^n$$

$$\partial W \to \partial D^n$$

$$S^n = D^n / \partial D^n$$

Suppose M_1^k and M_2^k are 2 such manifolds, each equipped with a framing in $\mathbb{R}^{n+k}$

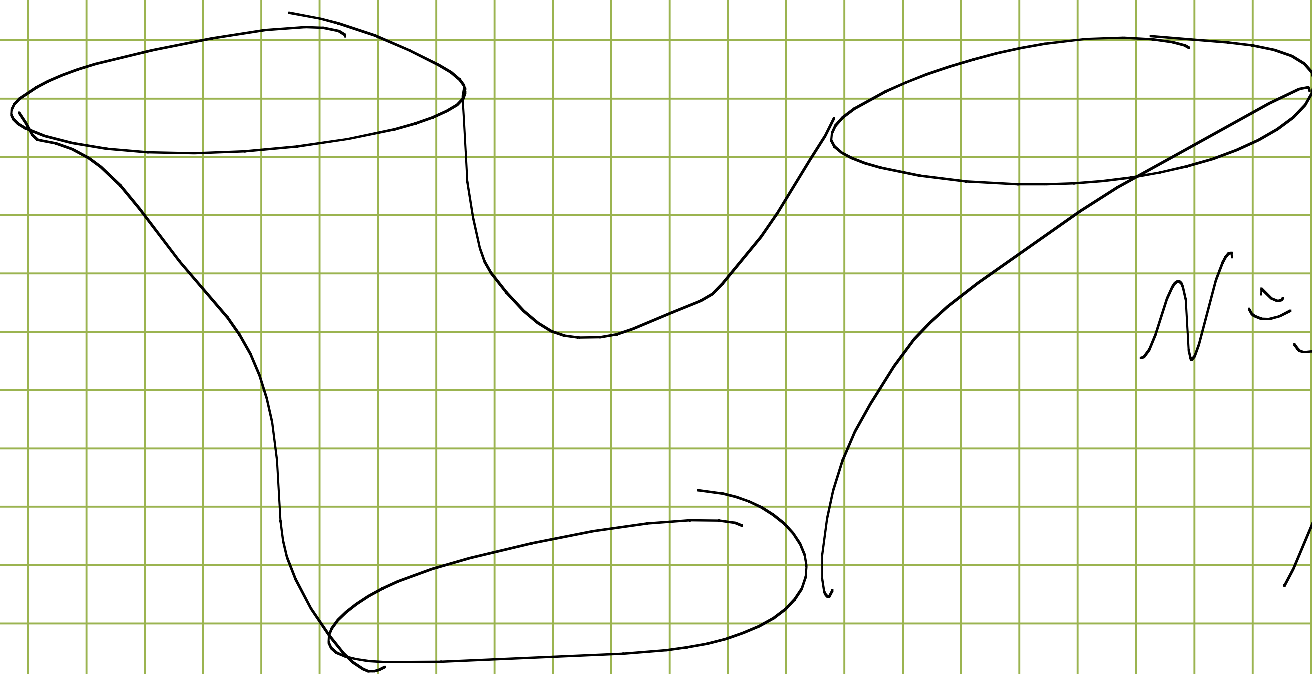
A cobordism between M_1 and M_2 is a mfd $N^{k+1} \subset \mathbb{R}^{n+k} \times [0,1]$ with

$$N \cap \mathbb{R}^{n+k} \times \{0\} = M_1 \qquad \partial N = M_1 \amalg M_2$$

$$N \cap \mathbb{R}^{n+k} \times \{1\} = M_2$$

N has a framing in $\mathbb{R}^{n+k} \times I$ restricting to the ones on M_1 and M_2.

A similar construction produces a map $S^{n+k} \times I \xrightarrow{h} S^n$, a homotopy between g_1 and g_2.



$M_1 = S^1 \amalg S^1$

N = surface

$M_2 = S^1$

This leads to an equivalence relation among framed mfds and a group (under disjoint union) of cobordism classes $\Omega^{fr}_{k,n}$

Thm $\Omega^{fr}_{k,n} \cong \pi_{n+k} S^n$

Pontryagin constructed hom

$$\Omega^{fr}_{k,n} \longrightarrow \pi_{n+k} S^n$$

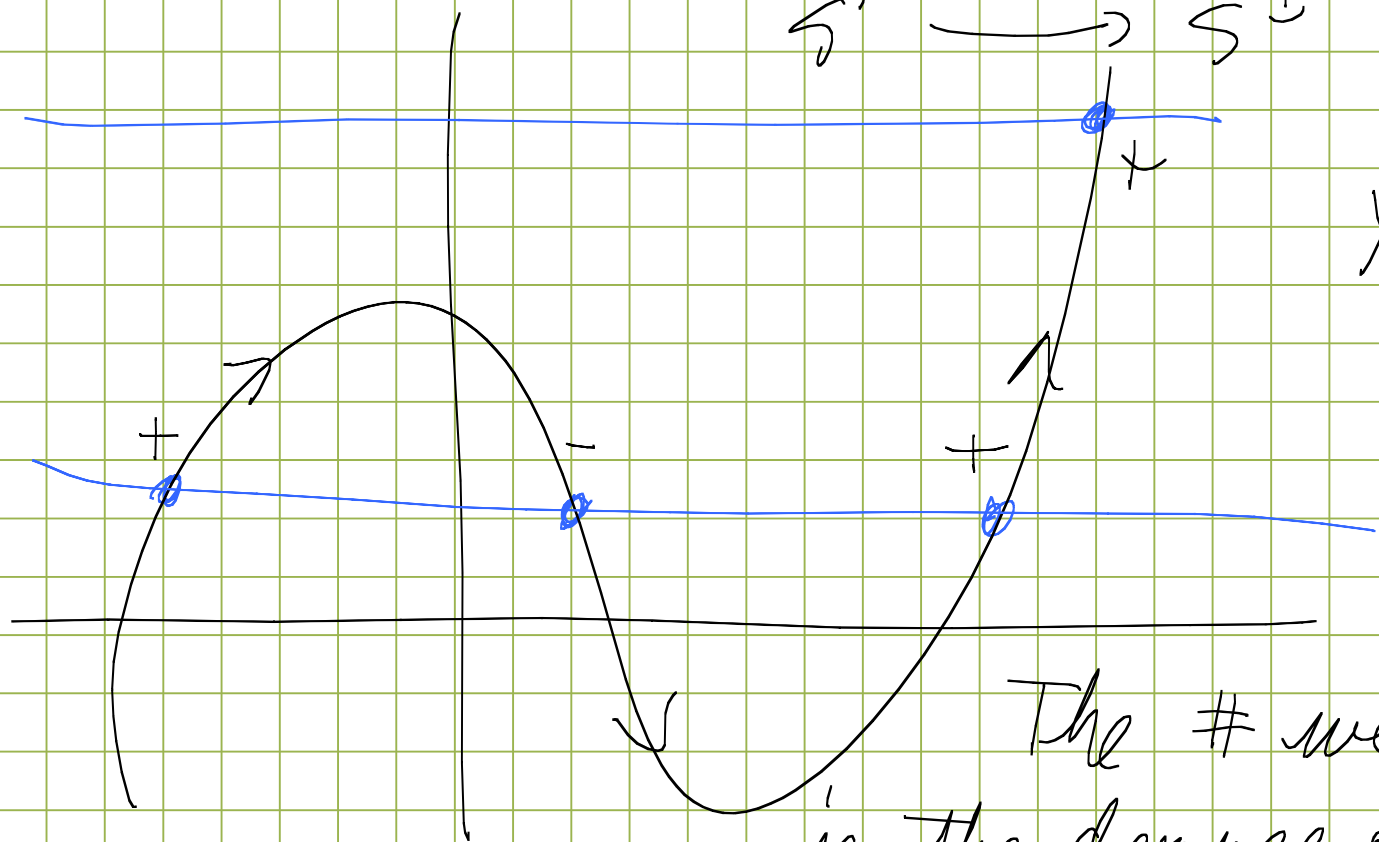
Thom showed it is an isomorphism

Group depends only on k for $n > k+1$.

$k=0$: M = finite set of points to be counted algebraically

$k=0 \quad n=1 \qquad \mathbb{R}^1 \xrightarrow{f} \mathbb{R}^1$

$S^1 \longrightarrow S^1$



The # we get is the degree of the map

$$\pi_n S^n = \mathbb{Z}$$

$k = 1$ M = union of circles

$n = 2$ e.g. $M = S^1 \subset \mathbb{R}^3$

It has a nbd $\cong S^1 \times D^2$ = solid torus

Projecting onto D^2 in the "obvious" way leads to a map $S^3 \xrightarrow{g} S^2$. The right framing extends to that of a D^2 in $\mathbb{R}^3 \times I$ and $g \simeq *$.

Two such framings differ by a map

$$S^1 \longrightarrow O(2) = 2\times2 \text{ orthogonal gp.}$$

$$\begin{aligned} &\quad \cong \\ &\mathbb{Z}/2 \times SO(2) \approx \mathbb{Z}/2 \times S^1 \end{aligned}$$

If the framings agree at one pt, we are mapping to S^1

$$\mathbb{Z} \simeq \pi_1(S^1) \longrightarrow \pi_3(S^2)$$

Hopf showed this map is an iso 1930

SIDEBAR on J-homomorphism

Let
$$\begin{tikzcd} S^k \arrow[r, phantom, "\subset"] & \mathbb{R}^{n+k} \arrow[r, squiggly] & S^{n+k} \arrow[r, "g"] & S^n \\ S^k \times D^n \arrow[u, phantom, "\cap"] \arrow[ur, bend right] & & & \end{tikzcd}$$

The naive choice leads to a null homotopic g.

We could alter this framing by an element of $\pi_k SO(n)$ and get a hom

$$\pi_k(SO(n)) \xrightarrow{J} \pi_{n+k}(S^n)$$

Hopf + GW Whitehead 1940's.

Bott determined $\pi_k SO(n)$ for $n > k+1$, $k > 0$.

and showed it is

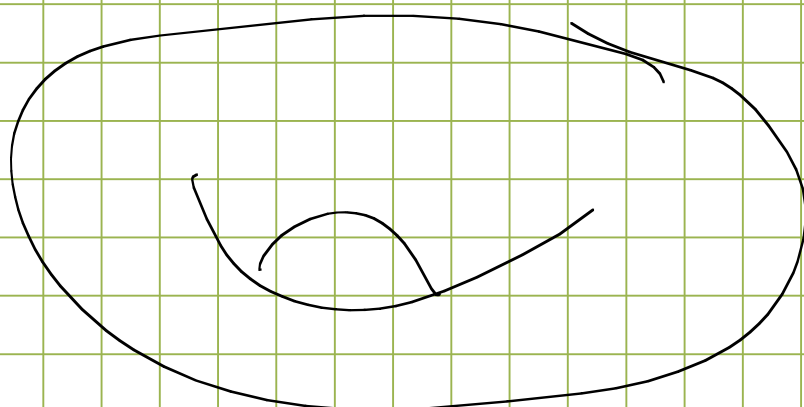
$$\begin{cases} \mathbb{Z}/2 & \text{for } k \equiv 0, 1 \mod 8 \\ \mathbb{Z} & \text{for } k \equiv 3, 7 \mod 8 \\ 0 & \text{otherwise} \end{cases}$$

END OF SIDEBAR

$k=2 \qquad M^2 = $ surface.

If it is S^2, then any framing leads to a null homotopic g.

Suppose $M = S^1 \times S^1 = $ torus $= T$



$n \geq 3$

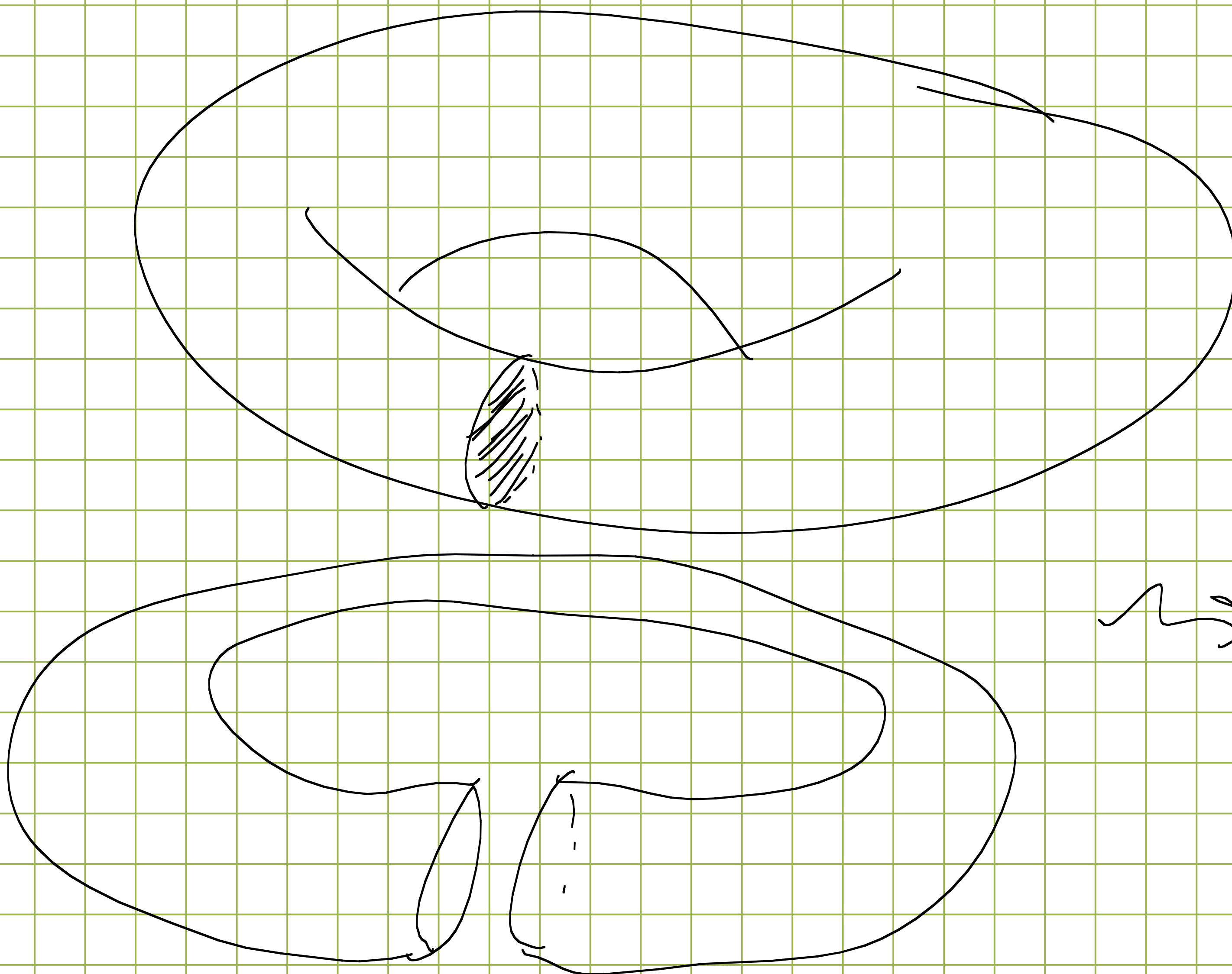
For each circle on T we get a framing related to $\pi_1 SO(n) = \mathbb{Z}/2$

This leads

$$\begin{tikzcd} H_1(M) \arrow[r] \arrow[dr] & \pi_1 SO(n) = \mathbb{Z}/2 \\ & H_1(M;\mathbb{Z}/2) \arrow[u] \end{tikzcd}$$

$$H_1(T;\mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2 \xrightarrow{\varphi} \mathbb{Z}/2$$

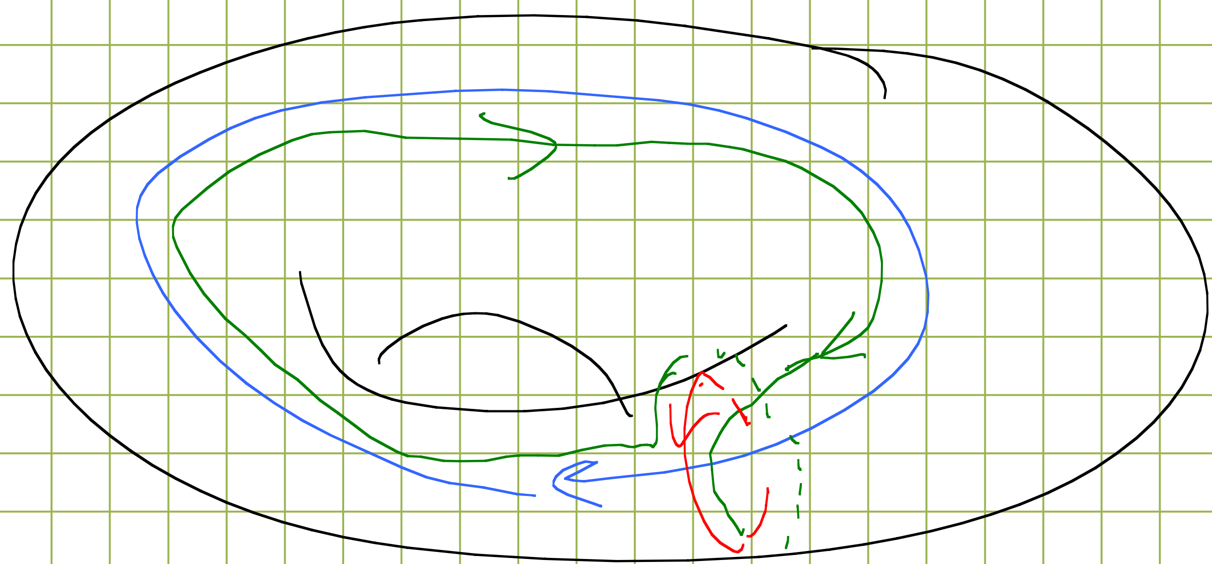
Choose a nontrivial elt in $\ker \varphi$. This corresponds to a circle in T along which the framing is untwisted, i.e. it extends to a D^2.



$\rightsquigarrow \pi_{n+2} S^n = 0$

add disks and get S^2

Pontryagin assumed incorrectly that ℓ is a homomorphism

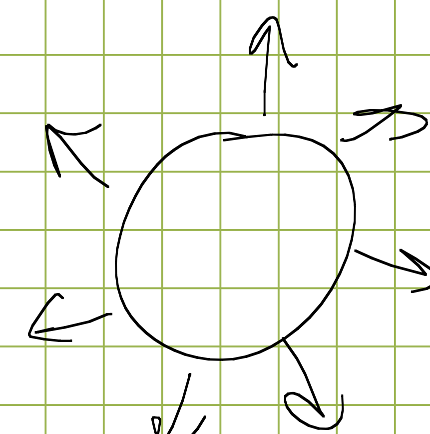


$H_1(T;\mathbb{Z}/2)$

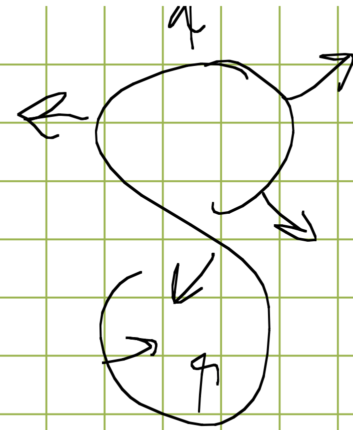
a, b

$a+b$

2 ways to frame S^1 in $\mathbb{R}^3$



If a and b have trivial framings, $a+b$ does not.



This means the map $\varphi : H_1(T; \mathbb{Z}/2) \to \mathbb{Z}/2$ is not a homomorphism!

What can we say about it?

$$\varphi(\alpha+\beta) = \varphi(\alpha) + \varphi(\beta) + \lambda(\alpha, \beta)$$

where λ is bilinear

$\lambda : H_1 \otimes H_1 \longrightarrow \mathbb{Z}/2$ hom

$q : H_1 \longrightarrow \mathbb{Z}/2$ not a hom

λ is related to intersections

Work of Arf 1941 Bilinear forms in characteristic 2

Suppose we have a nonsingular λ on a $\mathbb{Z}/2$-vector space H

$$H \otimes H \xrightarrow{\lambda} \mathbb{Z}/2 \qquad \dim H = 2g$$

suppose it is nonsingular, i.e.

Start with a skew symm bilinear over $\mathbb{Z}$ and reduce it mod 2 to get λ. Suppose further that there is a q as above.

We know we can choose a basis $\{a_i, b_i : 1 \le i \le g\}$ such

$$\lambda(a_i, b_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j. \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda(a_i, a_j) = \lambda(b_i, b_j) = 0$$

$$\begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & & \mathbf{0} \\ & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{bmatrix}$$ symplectic basis

$$\varphi(\alpha+\beta) = \varphi(\alpha) + \varphi(\beta) + \lambda(\alpha,\beta)$$

$$\operatorname{Arf}(\varphi) = \sum_{i=1}^{g} \varphi(a_i)\varphi(b_i) \in \mathbb{Z}/2$$

Arf showed this is a complete isomorphism invariant.

x	$\varphi(x)$	$\varphi'(x)$
0	0	0
a	0	1
b	0	1
$a+b$	1	1

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\mathrm{Anf} = 0$ $\qquad$ $\mathrm{Anf} = 1$