Let

$$G_k(\tau) := \sum_{m,n\in\mathbf{Z}}{}' \frac{1}{(m\tau+n)^k}$$

where the sum is over all nonzero lattice points. This vanishes if k is odd and is known to converge for $k > 2$. Note that

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{m,n\in\mathbf{Z}}{}' \left(\frac{c\tau+d}{m(a\tau+b)+n(c\tau+d)}\right)^k$$
$$= (c\tau+d)^k G_k(\tau).$$

Now let $q = e^{2\pi i\tau}$. In terms of it we have

$$G_k(\tau) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

where ζ is the Riemann zeta function, for which

$$\zeta(k) = (-2\pi i)^k \frac{B_k}{2k!} \qquad \text{for } k \text{ even},$$

B_k is the kth Bernoulli number defined by

$$\frac{x}{e^x-1} =: \sum_{k\geq 0} B_k \frac{x^k}{k!},$$

and

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

It is convenient to normalize G_k by defining the Eisenstein series

$$E_k(\tau) := \frac{1}{2\zeta(k)} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

It turns out that

$$E_k(\tau) = \sum_{(m,n)=1} \frac{1}{(m\tau+n)^k},$$

where the sum is over pairs of integers that are relatively prime.

Let

$$\begin{aligned} \Delta &:= \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} \\ &= q \prod_{n=1}^{\infty} (1-q^n)^{24} \end{aligned}$$

$$\text{and} \qquad j := E_4^3/\Delta.$$

Δ is called the discriminant, and the modular function j of weight 0 is a complex analytic isomorphism between H/Γ and the Riemann sphere.

It is known that the ring of all modular forms with respect to $\Gamma = SL_2(\mathbf{Z})$ is

$$M_*(\Gamma) = \mathbf{C}[E_4, E_6],$$

with (Δ) being the ideal of forms that vanish at $i\infty$, which are called *cusp forms*.

The *Weierstrass equation* for an elliptic curve in affine form is

(1) $y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6.$

Each a_k can be equated with a modular function of weight $2k$. The Eisenstein series are related to them by

$$E_4 = b_2^2 - 24b_4$$

and

$$E_6 = -b_2^3 + 36b_2b_4 - 216b_6$$

where

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= a_1a_3 + 2a_4 \\ b_6 &= a_3^2 + 4a_6. \end{aligned}$$

Then

$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$

is the discriminant of the Weierstrass equation, meaning that the curve is smooth iff Δ is a unit in the ground ring.

If 6 is invertible in the ground ring, then the Weierstrass equation can be rewritten as

$$y^2 = x^3 + g_2 x + g_3$$

Its modular forms are

$$\begin{aligned} E_4 &= -2^4 3 g_2 \\ E_6 &= -2^5 3^3 g_3 \\ \Delta &= -2^4(4g_2^3 + 27g_3^2) \\ j &= \frac{2^8 3^3 g_2^3}{4g_2^3 + 27g_3} \end{aligned}$$

Another special case of (1) is the *Legendre curve*,

$$y^2 = x(x-1)(x-\lambda),$$

for which the modular forms are

$$\begin{aligned} E_4 &= 2^4(1-\lambda+\lambda^2) \\ E_6 &= 2^5(\lambda-2)(\lambda+1)(2\lambda-1) \\ \Delta &= 2^4\lambda^2(\lambda-1)^2 \\ j &= \frac{2^8(1-\lambda+\lambda^2)^3}{(\lambda-1)^2\lambda^2} \end{aligned}$$

One also has the *Jacobi quartic*,

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4,$$

which is isomorphic to the Weierstrass curve

$$y^2 = (x - 12\delta)((x + 6\delta)^2 - 324\epsilon).$$

Its modular forms are

$$\begin{aligned} E_4 &= 2^6 3^4 (\delta^2 + \epsilon) \\ E_6 &= 2^9 3^6 \delta(\delta^2 - 9\epsilon) \\ \Delta &= 2^{12} 3^{12} (\epsilon - \delta^2)^2 \epsilon \\ j &= \frac{2^6(\delta^2 + 3\epsilon)^3}{\epsilon(\epsilon - \delta^2)^2}. \end{aligned}$$

A *formal group law* over a ring R is a power series

$$F(x,y) \in R[[x,y]]$$

satisfying

(i) $F(x,0) = F(0,x) = x$ (identity element)

(ii) $F(y,x) = F(x,y)$ (commutativity)

(iii) $F(x,F(y,z)) = F(F(x,y),z)$ (associativity)

Here are some examples

(a) $F(x,y) = x+y$, the additive formal group law.

(b) $F(x,y) = x+y+xy$, the multiplicative formal group law.

(c) $F(x,y) = \dfrac{x+y}{1+xy}$, the addition formula for tanh.

(d)

$$F(x,y) = \frac{x\sqrt{1-2\delta y^2+\epsilon y^4}+y\sqrt{1-2\delta x^2+\epsilon x^4}}{1-\epsilon x^2y^2},$$

the formal group law associated with the Jacobi quartic. It is defined over $\mathbf{Z}[1/2]$ and is due to Euler.

Isaac Newton Institute, Cambridge, UK