SIMPLE EXAMPLES OF LIMITS AND COLIMITS

MATH 549, FALL 2001

A model category $\mathcal{C}$ is assumed to have arbitrary limits and colimits. First we recall the relevant definition. We start with a functor $F\colon \mathcal{J} \to \mathcal{C}$ from a small category $\mathcal{J}$, i.e., a category in which the class of objects is a set. Such a functor should be thought of as a diagram in $\mathcal{C}$ with the shape of $\mathcal{J}$.

Here are some examples.

(i) Let $\mathcal{N}$ be the category in which there is one object $[n]$ for each natural number n and one morphism $[m] \to [n]$ for each $m \geq n$. Then a functor $F\colon \mathcal{N} \to \mathcal{C}$ is a diagram in $\mathcal{C}$ of the form

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

where $X_n = F([n])$. A functor from $\mathcal{N}^{\mathrm{op}}$ (the opposite category of $\mathcal{N}$) is a diagram

$$X_0 \to X_1 \to X_2 \to X_3 \to \cdots .$$

(ii) Let $\mathcal{D}$ be the category with three objects and two nonidentical morphisms as shown in the diagram

$$\begin{array}{ccccc} A_1 & & & & A_2 \\ & \searrow^{p_1} & & {}^{p_2}\swarrow & \\ & & A_0 & & \end{array}$$

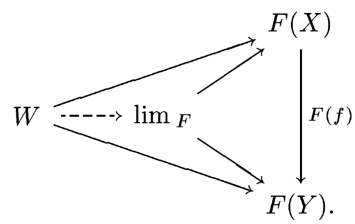
Then a functor $F\colon \mathcal{D} \to \mathcal{C}$ is a similar diagram in $\mathcal{C}$.

(iii) Let G be a group and $\mathcal{G}$ be the categrory with one object, denoted by G, and one morphism for each element $g \in G$. Composition of morphisms corresponds to group multiplication. Then a functor $F\colon \mathcal{G} \to \mathcal{C}$ is an action of the group G on the object $F(G)$ in $\mathcal{C}$.

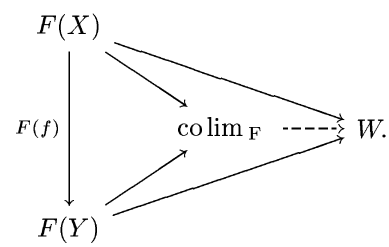
Now given a functor $F : \mathcal{J} \to \mathcal{C}$, the limit $\lim_F$ is an object in $\mathcal{C}$ equipped with a morphism to $F(j)$ for each object j in $\mathcal{J}$ such that for each morphism $f : X \to Y$ in $\mathcal{J}$ the following diagram commutes

$$\begin{array}{ccc} & & F(X) \\ & \nearrow & \\ \lim_F & & \downarrow{\scriptstyle F(f)} \\ & \searrow & \\ & & F(Y) \end{array}$$

Moreover $\lim_F$ has the following universal property. Given another object W in $\mathcal{C}$ with maps to the $F(j)$ satisfying similar conditions, such maps factor uniquely through $\lim_F$, as shown in the diagram



The colimit $\operatorname{co}\lim_F$ is defined similarly with the arrows not coming from morphisms in $\mathcal{J}$ reversed, so the diagram for the universal property is

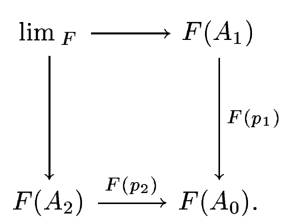


Now here are some examples:

(0) Consider the empty functor ϕ from the empty category to $\mathcal{C}$. Then the definition of $\lim_\phi$ requires it to be an object in $\mathcal{C}$ with a unique morphism from every other object in $\mathcal{C}$. This is called a *terminal object* in $\mathcal{C}$, and is usually denoted by 1 or pt.. Similarly $\operatorname{co}\lim_\phi$ is an *initial object* in $\mathcal{C}$, and is usually denoted by 0 or ϕ. For $\mathcal{C} =$ **Top** or **Sets**, the initial object is the empty set and the terminal object is a one point space or set.

(i) For $\mathcal{J} = \mathcal{N}$, $\lim_F$ is the inverse limit $\lim_{\leftarrow} X_i$, and for $\mathcal{J} = \mathcal{N}^{\mathrm{op}}$, $\operatorname{co}\lim_F$ is the direct limit $\lim_{\rightarrow} X_i$.

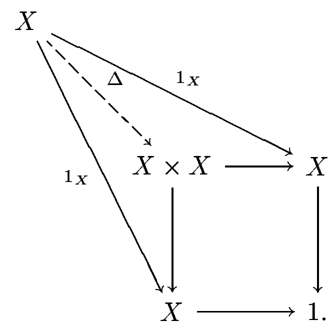
(ii) For $\mathcal{J} = \mathcal{D}$, $\lim_F$ is the *pullback* in the diagram



If $F(A_0)$ is the terminal object 1, then the pullback is the product of $F(A_1)$ and $F(A_2)$. If the objects of $\mathcal{C}$ have underlying sets, then the pullback can be described explicitly as

$$\{(x, y) \in F(A_1) \times F(A_2) \colon F(p_1)(x) = F(p_2)(y)\}$$

Suppose $F(A_0)$ is the terminal object 1 and $F(A_1) = F(A_2) = X$. Then the *diagonal map* $\Delta : X \to X \times X$ is the one given by the universal property for the diagram



Dually one gets the *fold map* $\nabla : X \coprod X \to X$ from the coproduct or disjoint union $X \coprod X$.