COFIBRANTLY GENERATED MODEL CATEGORIES

MATH 549, FALL 2001

This note is intended to summarize the statements in Hovey about the properties of cofibrantly generated model categories, culminating in Theorem 2.1.19. We will refer to numbered statements in his book and use his notation with the following exception. He routinely uses the symbols I and J to refer to the sets of generating cofibrations and trivial cofibrations respectively. We will instead denote them by $\mathcal{I}$ and $\mathcal{J}$.

Definition 1. *[Hovey 2.1.7] Let* $\mathcal{I}$ *be a class of maps in a category* $\mathbf{C}$*. A map* $f : X \to Y$ *in* $\mathbf{C}$ *is* *$\mathcal{I}$-injective* *if for each map* $i \in \mathcal{I}$ *the following commutative diagram always has the indicated lifting.*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

The class of such maps is denoted by $\mathcal{I}$*-inj.*

The map f *is an* *$\mathcal{I}$-cofibration* *if for each* $p \in \mathcal{I}$*-inj the following commutative diagram always has the indicated lifting.*

$$\begin{array}{ccc} X & \longrightarrow & K \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & L \end{array}$$

The class of such maps is denoted by $\mathcal{I}$*-cof.*

Definition 2. *[Hovey 2.1.9] Let the category* $\mathbf{C}$ *have small colimits, e.g. pushouts. A* *relative $\mathcal{I}$-cell complex* *is a transfinite composition of pushouts of maps in* $\mathcal{I}$*, i.e. of maps of the form* $X \to Y$ *in the pushout diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

with $i \in \mathcal{I}$*. The class of such maps is denoted by* $\mathcal{I}$*-cell.*

Theorem 3. *[Hovey 2.1.4, the small object argument] With* $\mathbf{C}$ *and* $\mathcal{I}$ *as above, suppose that the domains of* $\mathcal{I}$ *are all small relative to* $\mathcal{I}$*-cell. Then for any map* $f : X \to Y$ *in* $\mathbf{C}$ *there is a functorial factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & T \\ & {\scriptstyle \gamma(f)} \searrow \quad \nearrow {\scriptstyle \delta(t)} & \\ & X' & \end{array}$$

where $\gamma(f) \in \mathcal{I}$*-cell and* $\delta(f) \in \mathcal{I}$*-inj.*

Definition 4. *[Hovey 2.1.17] A model category* $\mathbf{C}$ *is* <u>*cofibrantly generated*</u> *if there are set of maps* $\mathcal{I}$ *and* $\mathcal{J}$ *such that*

(i) *the domains of* $\mathcal{I}$ *are small relative to* $\mathcal{I}$*-cell,*
(ii) *the domains of* $\mathcal{J}$ *are small relative to* $\mathcal{J}$*-cell,*
(iii) *the class of fibrations is* $\mathcal{J}$*-inj, and*
(iv) *the class of trivial fibrations is* $\mathcal{I}$*-inj.*

$\mathcal{I}$ is called the <u>set of generating cofibrations</u> and $\mathcal{J}$ is called the <u>set of generating trivial cofibrations</u>.

Proposition 5. *[Hovey 2.1.18] Let* $\mathbf{C}$*,* $\mathcal{I}$ *and* $\mathcal{J}$ *be as above. Then*

(a) *The cofibrations form the class* $\mathcal{I}$*-cof.*
(b) *Every cofibration is a retract of a relative* $\mathcal{I}$*-cell complex.*
(c) *The domains of* $\mathcal{I}$ *are small relative to the cofibrations.*
(d) *The trivial cofibrations form the class* $\mathcal{J}$*-cof.*
(e) *Every trivial cofibration is a retract of a relative* $\mathcal{J}$*-cell complex.*
(f) *The domains of* $\mathcal{J}$ *are small relative to the trivial cofibrations.*

Now we can state the main result of §2.1 and the main tool for constructing model categories.

Theorem 6. *[Hovey 2.1.19] Let* $\mathbf{C}$ *be a category with all small limits and colimits. Suppose it has a subcategory* W *and sets of maps* $\mathcal{I}$ *and* $\mathcal{J}$*. Then there is a cofibrantly generated model structure on* $\mathbf{C}$ *with fibrations and trivial fibrations as above with with weak equivalence* $\mathbf{W}$ *if and only the following conditions are satisfied.*

(i) *The subcategory* $\mathbf{W}$ *has the two out of three property and is closed under retracts.*
(ii) *The domains of* $\mathcal{I}$ *are small relative to* $\mathcal{I}$*-cell.*
(iii) *The domains of* $\mathcal{J}$ *are small relative to* $\mathcal{J}$*-cell.*
(iv) $\mathcal{J}$*-cell* $\subseteq \mathbf{W} \cap \mathcal{I}$*-cof ('trivial cofibrations' really are cofibrations that are weak equivalences).*
(v) $\mathcal{I}$*-inj* $\subseteq \mathbf{W} \cap \mathcal{J}$*-inj ('trivial fibrations' really are fibrations that are weak equivalences).*
(vi) *Either* $\mathbf{W} \cap \mathcal{I}$*-cof* $\subseteq \mathcal{J}$*-cof or* $\mathbf{W} \cap \mathcal{J}$*-inj* $\subseteq \mathcal{I}$*-inj. (This is needed to prove the two lifting axioms.)*

Establishing a model structure on a category $\mathbf{C}$ amounts to defining $\mathbf{W}$, $\mathcal{I}$ and $\mathcal{J}$ and showing that they satisfy the conditions of the theorem.

The classical example of a cofibrantly generated model category is **Top** with the weak equivalences being the maps inducing isomorphisms in homotopy groups with respect to any base point, and $\mathcal{I}$ and $\mathcal{J}$ being the sets of standard inclusions

$$\begin{aligned} \mathcal{I} &= \left\{S^{n-1} \to D^n \colon n \geq 0\right\} \\ \text{and} \qquad \mathcal{J} &= \left\{D^n \to D^n \times I \colon n \geq 0\right\}. \end{aligned}$$