THE AXIOMS OF A MODEL CATEGORY

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This is an illustration of Definition 1.1.3 of Hovey.

A *model structure* on a catgeory $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, fibrations and cofibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties:

1. (2-out-of-3) Given a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{gf} & X \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Y & \end{array}$$

   if any two of the three maps shown is a weak equivalence, then so is the third.

2. (Retracts) Suppose f is a retract of g, i.e., there is a diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

   in which both horizontal composites are identities. If g is a weak equivalence, a fibration, or a cofibration, then so is f.

3. (Lifting) The indicated liftings always exist.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle \textit{trivial cofibration}}\downarrow & \nearrow & \downarrow{\scriptstyle \textit{fibration}} \\ B & \xrightarrow[g]{} & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle \textit{cofibration}}\downarrow & \nearrow & \downarrow{\scriptstyle \textit{trivial fibration}} \\ B & \xrightarrow[g]{} & Y \end{array}$$

   A fibration or cofibration is said to be *trivial* if it is also a weak equivalence. The maps f and g are arbitrary.

4. (Factorization) Any map f can be functorially factored in two ways as follows.

$$\begin{array}{ccc} & ? & \\ \textit{cofibration } \alpha(f) \nearrow & & \searrow \beta(f) \textit{ trivial fibration} \\ X & \xrightarrow{f} & Y \\ \textit{trivial cofibration } \gamma(f) \searrow & & \nearrow \delta(f) \textit{ fibration} \\ & ? & \end{array}$$