THE STRUCTURE OF $SO(4)$
MATH 443
APRIL 2, 2002

This is the solution to the extra credit problem due on March 19, namely a proof that $SO(4)$ is homeomorphic to $SO(3) \times S^3$.

For general n one has a map $p : SO(n) \to S^{n-1}$ obtained by evaluating an orthogonal matrix $g \in SO(n)$ on a fixed unit vector $x_0 \in S^{n-1}$. We claim that the preimage of each point under p is homeomorphic to $SO(n-1)$. First note that $p^{-1}(x_0)$ is the set of matrices which fix the subspace spanned by x_0, so it is the group of special orthogonal matrices acting on the hyperplane orthogonal to x_0.

More generally two elements in $p^{-1}(x)$ differ by an element of $SO(n)$ that fixes x, i.e., by a special orthogonal matrix acting on the hyperplane orthogonal to x. If we pick an element $g_x \in p^{-1}(x)$, then for any $g \in p^{-1}(x)$, $g_x^{-1}g$ is in the subgroup of $G_x \subset SO(n)$ that fixes x, and G_x is isomorphic to $SO(n-1)$.

Now suppose the map p has a lifting, i.e., a map $\ell : S^{n-1} \to SO(n)$ such that $p\ell$ is the identity map on S^{n-1}. Then arguments similar to those of 6.14 show that $SO(n)$ is homeomorphic to $S^{n-1} \times SO(n-1)$. It turns out that ℓ exists only if $n = 2$, 4 or 8.

The lifting for $n = 4$ is based on the structure of the *quaternions* $\mathbf{H}$ (see `http://mathworld.wolfram.com/Quaternion.html`). This a noncomuutative division algebra homeomorphic to $\mathbf{R}^4$. Mutiplication by any nonzero unit vector induces a special orthogonal transformation of $\mathbf{R}^4$, so we get a group structure on S^3 and a homomorphism $S^3 \to SO(4)$, which is the desired lift.

The lifting for $n = 8$ follows in a similar way from the existence of the *Cayley numbers* (see `http://mathworld.wolfram.com/CayleyAlgebra.html`), a nonassociative division algebra homeomorphic to $\mathbf{R}^8$.

The nonexistence of liftings for other values of n is a deeper theorem in algebraic topology. A good historical reference for it is May's paper

`http://www.math.uiuc.edu/K-theory/0321/`

starting on page 15.