HINTS ON HOMOLOGICAL ALGEBRA
MATH 443
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This note is intended to be a quick summary of some basic definitions and facts in homological algebra. More details can be found in any book on the subject.

The *tensor product* $M \otimes_R N$ of R-modules M and N is the R-module generated by symbols $m \otimes n$ for $m \in M$ and $n \in N$, subject to the relations

(i) $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$
(ii) $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$
(iii) $rm \otimes n = m \otimes rm$ for $r \in R$.

In particular for $R = \mathbf{Z}$ this defines the tensor product of two abelian groups. If $S \subset R$ is a subring then M and N are S-modules and one can define the S-module $M \otimes_S N$ in a similar way. As an S-module, $M \otimes_R N$ is a quotient of $M \otimes_S N$.

It follows that

$$\begin{aligned}
&M \otimes_R (N_1 \oplus N_2) = M \otimes_R N_1 \oplus M \otimes_R N_2 \\
&(M_1 \oplus M_2) \otimes_R N = M_1 \otimes_R N \oplus M_2 \otimes_R N \\
&R \otimes_R M = M \otimes_R R = M.
\end{aligned}$$

Tensoring an exact sequence of R-modules with a fixed R-module N does not in general preserve exactness, and the functor Tor measures its failure to do so. Given an R-module M we can always find a long exact sequence (called a free resolution)

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0 \tag{1}$$

where each F_i is a free R-module. (One could substitute projective modules for free ones.) If R is a principal ideal domain such as Z, then this resolution can always be chosen for that $F_i = 0$ for $i > 1$, i.e., there is always a short exact sequence

$$0 \to F_1 \to F_0 \to M \to 0.$$

In any case tensoring (1) with N gives us a chain complex of the form

$$\cdots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N, \tag{2}$$

and its ith homology group is denoted by $\mathrm{Tor}_i^R(M,N)$, the *Tor group*. It is known to be independent of the choice of resolution. Tor is functorial on both variables and it is symmetric, i.e., $\mathrm{Tor}_i^R(M,N)$ is isomorphic to $\mathrm{Tor}_i^R(N,M)$. $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$ and the higher Tor groups vanish if either M or N is free. When $R = \mathbf{Z}$, $\mathrm{Tor}_1^{\mathbf{Z}}(M,N)$ is denoted simply by $\mathrm{Tor}(M,N)$.

A short exact sequences in either variable leads to a short exact of the chain complexes in (2) and hence to a long exact sequence of Tor groups. For example given a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

we get a long exact sequence

$$\begin{array}{ccccccc}
 & & & & & & \cdots \\
\operatorname{Tor}_2^R(M', N) & \longrightarrow & \operatorname{Tor}_2^R(M, N) & \longrightarrow & \operatorname{Tor}_2^R(M'', N) & & \\
\operatorname{Tor}_1^R(M', N) & \longrightarrow & \operatorname{Tor}_1^R(M, N) & \longrightarrow & \operatorname{Tor}_1^R(M'', N) & & \\
M' \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & M'' \otimes_R N & \longrightarrow & 0.
\end{array}$$

When $R = \mathbf{Z}$ this reduces to a six term exact sequence.

Similarly applying the functor $\operatorname{Hom}_R(\cdot, N)$ (the group of R-module homomorphisms $M \to N$, which is itself an R-module) to an exact sequence does not preserve exactness, and the functor Ext measures its failure to do so. Applying it to (1) leads to a cochain complex

$$\operatorname{Hom}_R(F_0, N) \to \operatorname{Hom}_R(F_1, N) \to \operatorname{Hom}_R(F_2, N) \to \cdots \tag{3}$$

whose ith cohomology group is denoted by $\operatorname{Ext}_R^i(M, N)$. It is also known to be independnet of the choice of resolution. Unlike Tor, it is contravariant (arrow reversing) on the first variable. We have $\operatorname{Ext}_R^0(M, N) = \operatorname{Hom}_R(M, N)$ and the higher Ext groups vanish if M is free, but not necessarily if N is free. When $R = \mathbf{Z}$, $\operatorname{Ext}_{\mathbf{Z}}^i(M, N) = 0$ for $i > 1$ and $\operatorname{Ext}_1^{\mathbf{Z}}(M, N)$ is denoted simply by $\operatorname{Ext}(M, N)$.

Short exact sequences in either variable lead to long exact sequnces of Ext groups. For example

$$0 \to M' \to M \to M'' \to 0$$

leads to a long exact sequence

$$\begin{array}{ccccccc}
 & & & & & & \cdots \\
\operatorname{Ext}_R^2(M', N) & \longleftarrow & \operatorname{Ext}_R^2(M, N) & \longleftarrow & \operatorname{Ext}_R^2(M'', N) & & \\
\operatorname{Ext}_R^1(M', N) & \longleftarrow & \operatorname{Ext}_R^1(M, N) & \longleftarrow & \operatorname{Ext}_R^1(M'', N) & & \\
\operatorname{Hom}_R(M', N) & \longleftarrow & \operatorname{Hom}_R(M, N) & \longleftarrow & \operatorname{Hom}_R(M'', N) & \longleftarrow & 0.
\end{array}$$

When $R = \mathbf{Z}$ this reduces to a six term exact sequence as in the case of Tor.