A VARIANT OF THE FIVE LEMMA
MATH 443
APRIL 2, 2002

This is the solution to the extra credit problem concerning a variant of the Five lemma (14.7) in which the conclusion is that the middle map is trivial. Here it is.

Lemma 1. *Suppose we have a commutative diagram of R-modules with exact rows.*

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \\
\downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \epsilon'} \\
C_1 & \xrightarrow{h_1} & C_2 & \xrightarrow{h_2} & C_3 & \xrightarrow{h_3} & C_4 & \xrightarrow{h_4} & C_5
\end{array}$$

If δ and β' are both trivial, then so is the composite $\gamma'\gamma$.

In particular the two outer columns of the diagram are irrelevant, and exactness is only needed at B_3.

Proof. Let x be an element of A_3. Then

$$g_3\gamma(x) = \delta f_3(x)$$

so $\gamma(x) = g_2(y)$ for some $y \in B_2$. Hence

$$\begin{aligned}
\gamma'\gamma(x) &= \gamma' g_2(y) \\
&= h_2\beta'(y) \\
&= h_2(0) \\
&= 0. \quad \square
\end{aligned}$$