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# $\pi_*(L_2T(1)/(v_1))$ and its applications in computing $\pi_*(L_2T(1))$ at the prime two

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**Abstract.** In this paper, all spaces are localized at the prime two. Let $T(1)$ be the Ravenel spectrum characterized by the $BP_*$-homology as $BP_*[t_1]$, $T(1)/(v_1)$ be the cofiber of the self map $v_1 : \Sigma^2 T(1) \to T(1)$ and $L_2$ denote the Bousfield localization functor with respect to $v_2^{-1}BP_*$. In this paper, we compute the homotopy groups $\pi_*(L_2T(1)/(v_1))$ by determining the $E_\infty$-term of its Adams-Novikov spectral sequence (ANSS). From the $E_2$-term of the ANSS for $\pi_*(L_2T(1)/(v_1))$, we determine a subgroup of the $E_2$-term for $\pi_*(L_2T(1))$. We also show that the $E_4$-term for $\pi_*(L_2T(1))$ has horizontal vanishing line.

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## 1 Introduction

In this paper, all spaces are localized at the prime two. Let $T(1)$ be the Ravenel spectrum characterized by the $BP_*$ homology as

$$BP_*(T(1)) = BP_*[t_1] \subset BP_*BP$$

(*cf.* [12]). Let $L_2$ denote the Bousfield localization functor with respect to $v_2^{-1}BP_*$ (see [11]). One of the methods to determine the homotopy groups $\pi_*(L_2T(1))$ is the Adams-Novikov spectral sequence with $E_2^*(L_2T(1)) = H^*v_2^{-1}BP_*[t_1] \Rightarrow \pi_*(L_2T(1))$, where $H^*- = Ext^*_{BP_*BP}(BP_*, -)$. We compute the $E_2$-term $E_2^*(L_2T(1)) = H^*v_2^{-1}BP_*[t_1]$ by the chromatic spectral sequence $\sum_{i=0}^{2} H^*M_0^i[t_1] \Rightarrow H^*v_2^{-1}BP_*[t_1]$ and the Bockstein spectral sequence $H^*M_{i+1}^{j-1}[t_1] \Rightarrow H^*M_i^j[t_1]$. (Note here all of the algebraic constructions have geometric origin) (*cf.* [8], [12, Ch. 5]).

In [12, 6.5.7] it is shown that $H^*M_0^0[t_1] = K(0)_*[v_1]$ and $H^*M_1^0[t_1] = K(1)_*[v_2] \otimes \Lambda(h_{20})$, where $K(0)_* = \mathbf{Q}$ and $K(1)_* = Z/2[v_1^{\pm 1}]$. Then, by the 2-Bockstein spectral sequence $H^*M_1^0[t_1] \Rightarrow H^*M_0^1[t_1]$, I. Ichigi and K. Shimomura determined $H^*M_0^1[t_1]$ in [3] to be:

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$$(1.1)\qquad H^*M_0^1[t_1] = Z_{(2)}[v_1^{\pm 1}] \otimes \left(\mathbf{Q}/Z_{(2)} \oplus \sum_{n,s\geq 0} Z/(2^{n+1})\langle v_2^{2^n(2s+1)}/2^{n+1}\rangle\right).$$

To determine the general chromatic $E_1$-term $H^*M_0^2[t_1]$, we have two ways to work back to $H^*M_0^2[t_1]$ through the Bockstein spectral sequences both started from $H^*M_2^0[t_1]$

$$H^*M_2^0[t_1] \Rightarrow H^*M_1^1[t_1] \Rightarrow H^*M_0^2[t_1],$$

$$H^*M_2^0[t_1] \Rightarrow H^*L_1^1[t_1] \Rightarrow H^*M_0^2[t_1].$$

Here $L_1^1[t_1] = v_2^{-1}BP_*[t_1]/(2^\infty, v_1)$.

In [6], M. Mahowald and K. Shimomura determined $H^*M_2^0[t_1]$ as:

$$H^*M_2^0[t_1] = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2)$$

where $K(2)_* = Z/2[v_2^{\pm 1}]$. From which K. Shimomura in [13] determined $H^*M_1^1[t_1]$ to be the tensor product of $\Lambda(\rho_2)$ and the direct sum of modules $A_i$:

$$A_0 = (v_1^{-1}K/K \oplus A_0^+ \oplus A_0^-) \otimes \Lambda(\widetilde{h_{20}})$$

$$A_1 = v_3^2K/(v_1^2)[x_1] \otimes \Lambda(h_{30}, h_{31})$$

$$A_2 = v_3K(2)_*[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}),$$

where

$$A_0^+ = \sum_{n>0} x_nK/(v_1^{a_n})[x_n^2] \otimes \Lambda(\tilde{g}_n^2)$$

$$A_0^- = \sum_{n>0} x_n^2K/(v_1^{2a_n})[x_{n+1}] \otimes \Lambda(\tilde{g}_{n+1}).$$

Here $K = Z/2[v_1, v_2^{\pm 1}]$, $a_n = 4^n + 2(4^n - 1)/3$ and the elements $\widetilde{h_{20}}$, $x_n$ and $\tilde{g}_n$ denote the ones represented by the cocycles whose leading terms are $v_3^2t_2$, $v_3^{4^n}$ and $v_3^{4(4^{n-1}-1)/3}t_3$ respectively. But it seems very complicated to determine $H^*M_0^2[t_1]$ by the 2-Bockstein spectral sequence $H^*M_1^1[t_1] \Rightarrow H^*M_0^2[t_1]$ (*cf.* [4, 10]).

Let $T(1)/(2^\infty, v_1)$ be the cofiber of the localization map $T(1)/(v_1) \to L_0T(1)/(v_1)$, where the $T(1)/(v_1)$ is the cofiber of the $v_1$-map $v_1 : \Sigma^2T(1) \to T(1)$. Then $H^*L_1^1[t_1]$ is the $E_2$-term of the Adams-Novikov spectral sequence for $\pi_*(L_2T(1)/(2^\infty, v_1))$. In this paper, we will compute $H^*L_1^1[t_1]$. We also compute the Adams-Novikov differentials, and from which, we get the homotopy groups $\pi_*(L_2T(1)/(v_1))$ (Theorem 4.7) by the cofiber sequence

$$L_2T(1)/(v_1) \to L_0T(1)/(v_1) \to L_2T(1)/(2^\infty, v_1).$$

Consider the maps $1/v_1 : L_1^1[t_1] \to M_0^2[t_1]$ and $1/2 : M_1^1[t_1] \to M_0^2[t_1]$. Then from the special properties of $H^*M_0^2[t_1]$, we will see that they induce the following commutative diagrams:

$$(1.2)\qquad \begin{array}{ccccccc} H^sL_1^1[t_1] & \xrightarrow{\delta_1} & H^{s+1}M_2^0[t_1] & & H^sM_1^1[t_1] & \xrightarrow{\delta_2} & H^{s+1}M_2^0[t_1] \\ \downarrow{\scriptstyle 1/v_1} & & \downarrow{\scriptstyle 1/v_1} & & \downarrow{\scriptstyle 1/2} & & \downarrow{\scriptstyle 1/2} \\ H^sM_0^2[t_1] & \xrightarrow{\delta_1} & H^{s+1}M_1^1[t_1] & & H^sM_0^2[t_1] & \xrightarrow{\delta_2} & H^{s+1}L_1^1[t_1]. \end{array}$$

Define the homomorphisms $\Delta_1$ and $\Delta_2$ as:

$$(1.3)\qquad \begin{aligned} \Delta_1 &= 1/v_1 \cdot \delta_1 : H^sL_1^1[t_1] \to H^{s+1}M_2^0[t_1] \to H^{s+1}M_1^1[t_1] \\ \Delta_2 &= 1/2 \cdot \delta_2 : H^sM_1^1[t_1] \to H^{s+1}M_2^0[t_1] \to H^{s+1}L_1^1[t_1]. \end{aligned}$$

**Lemma 1.4.** *For the homomorphisms $\Delta_1$ and $\Delta_2$, we have:*

1. *For an element $x \in H^*L_1^1[t_1]$, $x/v_1 \neq 0 \in H^*M_0^2[t_1]$ if $\Delta_1(x) \neq 0 \in H^*M_1^1[t_1]$.*
2. *For an element $y \in H^*M_1^1[t_1]$, $y/2 \neq 0 \in H^*M_0^2[t_1]$ if $\Delta_2(y) \neq 0 \in H^*L_1^1[t_1]$.*

From Lemma 1.4 and $H^*L_1^1[t_1]$, we get the submodule of $H^*L_1^1[t_1] : \widetilde{A_2} = (\widetilde{A_{20}} \otimes \Lambda(\rho_2)) \oplus \widetilde{A_{21}}$ such that $ker\,\Delta_1 \cap ((\widetilde{A_{20}} \otimes \Lambda(\rho_2)) \oplus \widetilde{A_{21}}) = 0$, where

$$(1.5)\qquad \widetilde{A_{20}} = v_2v_3/2 \cdot K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}) \oplus v_3h_{31}/2 \cdot K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30})$$

$K_*^2 = Z/2[v_2^{\pm 2}]$ and $\widetilde{A_{21}}$ is given in (5.3).

**Theorem 1.6.** *The homomorphism $1/v_1 : (\widetilde{A_{20}} \otimes \Lambda(\rho_2))^s \to H^sM_0^2[t_1]$ is an isomorphism for $s > 3$.*

Let $T(1)/(2^\infty)$ be the cofiber of the localization map $T(1) \to L_0T(1)$ and $T(1)/(2^\infty, v_1^\infty)$ be the cofiber of the localization map $T(1)/(2^\infty) \to L_1T(1)/(2^\infty)$. Then $H^*M_0^2[t_1]$ is the $E_2$-term of the Adams-Novikov spectral sequence for $\pi_*(L_2T(1)/(2^\infty, v_1^\infty))$. In this paper, we also compute the Adams-Novikov differentials on $\widetilde{A_2}$ and then show that:

**Theorem 1.7.** *The $E_4$-term $E_4^{s,*}$ of the Adams-Novikov spectral sequence for computing $\pi_*(L_2T(1)/(2^\infty, v_1^\infty))$ is zero for $s > 5$. Thus it collapses from the $E_6$-term.*

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## 2 The self map $v_1 : \Sigma^2 T(1) \to T(1)$ and relations in $\Sigma(2)$

Note the change of rings theorem [9, 12]

$$E_2^{*,*} = Ext^{*,*}_{BP_*BP}(BP_*, BP_*T(1)) = Ext^{*,*}_{BP_*[t_2,\ldots]}(BP_*, BP_*).$$

Since $|t_2| = 6$, we see that $E_2^{s,t} = 0$ for $t - s = 1$.

**Lemma 2.1** (K. Shimomura). *There exists a self map $v_1 : \Sigma^2 T(1) \to T(1)$ such that $BP_*(v_1) = v_1$.*

*Proof.* Note that $d_r(v_1) \in E_r^{r,r+1}$, which is zero. We have the map $v_1' : S^2 \to T(1)$ that induces $v_1$ on $BP_*$-homology. Since $T(1)$ is a ring spectrum, we got the desired self map. □

Let $F(2)_*$ denote the $BP_*$-module $Z_{(2)}[v_1, v_2^{\pm 1}, v_3]$ with action given by sending $v_i$ to 0 for $i > 3$. Let $F(2)$ be the spectrum that represents the Landweber homology theory $F(2)_*(X) = F(2)_* \otimes_{BP_*} BP_*(X)$. Then $F(2)_*F(2) = F(2)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} F(2)_*$, and the Hopf algebroid structure of $(BP_*, BP_*BP)$ defines the one on $(F(2)_*, \Sigma) = (F(2)_*, F(2)_*F(2))$. In the same manner as the change of rings theorem $Ext^s_{BP_*BP}(BP_*, M) = Ext^s_{E(2)_*E(2)}(E(2)_*, E(2)_* \otimes_{BP_*} M)$ for $v_2$-local $BP_*$-comodule $M$, we obtain

$$Ext^s_\Sigma(F(2)_*, F(2)_* \otimes_{BP_*} M) = Ext^s_{BP_*BP}(BP_*, M).$$

For the $v_2$-local $BP_*$-comodule $M[t_1]$, we have $M[t_1] = M \square_{\Gamma(2)} BP_*BP$, where $\Gamma(2) = BP_*[t_2, t_3, \ldots]$, and

$$F(2)_* \otimes_{BP_*} M[t_1] = M \square_{\Sigma(2)} F(2)_*F(2)$$

for $\Sigma(2) = F(2)_*[t_2, t_3, \ldots]$. Thus we get the change of rings theorem

$$Ext^s_{BP_*BP}(BP_*, M[t_1]) = Ext^s_{\Sigma(2)}(F(2)_*, M)$$

and the Hopf algebroid structure of $(BP_*, \Gamma(2))$ determines the one on $(F(2)_*, \Sigma(2))$. In this paper, by the modules $H^*M[t_1]$, we mean to work on $Ext^*_{\Sigma(2)}(F(2)_*, M)$.

For the structure map $\eta_R : BP_* \to \Gamma(2)$ and $\Delta : \Gamma(2) \to \Gamma(2) \otimes \Gamma(2)$, it is easy to show that:

(2.2)
$$\eta_R(v_1) = v_1$$

$$\eta_R(v_2) = v_2 + 2t_2$$

$$\eta_R(v_3) = v_3 + v_1 t_2^2 - v_1^4 t_2 + 2(t_3 - v_1 v_2 t_2 - v_1 t_2^2)$$

$$\eta_R(v_4) = v_4 + v_2 t_2^4 + v_2^4 t_2 + v_1 t_3^2 \qquad \mathrm{mod}(2, v_1^2)$$

$$\eta_R(v_5) = v_5 + v_2 t_3^4 + v_2^8 t_3 + v_3 t_2^8 + v_3^4 t_2 \qquad \mathrm{mod}(2, v_1)$$

$$\begin{aligned}
\eta_R(v_6) &= v_6 + v_2t_4^4 + v_2^{16}t_4 + v_3t_3^8 + v_3^8t_3 \\
&\quad + v_4t_2^{16} + v_4^4t_2 + v_2t_2^{10} + v_2^{17}t_2^4 && \mathrm{mod}(2, v_1) \\
\Delta(t_i) &= t_i \otimes 1 + 1 \otimes t_i && \text{for } i = 1, 2 \\
\Delta(t_3) &= t_3 \otimes 1 + 1 \otimes t_3 + v_1t_2 \otimes t_2 \\
\Delta(t_4) &= t_4 \otimes 1 + t_2 \otimes t_2^4 + 1 \otimes t_4 + v_2t_2^2 \otimes t_2^2 + v_1t_3 \otimes t_3 && \mathrm{mod}(2, v_1^2) \\
\Delta(t_5) &= t_5 \otimes 1 + t_3 \otimes t_2^8 + t_2 \otimes t_3^4 + 1 \otimes t_5 \\
&\quad + v_2t_3^2 \otimes t_3^2 + v_3t_2^4 \otimes t_2^4 && \mathrm{mod}(2, v_1).
\end{aligned}$$

Thus in $\Sigma(2)$, we have

$$\begin{aligned}
(2.3)\qquad t_2^4 &= t_2 && \mathrm{mod}(2, v_1, v_2 - 1) \\
t_3^4 &= t_3 + v_3t_2^2 + v_3^4t_2 && \mathrm{mod}(2, v_1, v_2 - 1) \\
t_4^4 &= t_4 + v_3t_3^8 + v_3^8t_3 && \mathrm{mod}(2, v_1, v_2 - 1).
\end{aligned}$$

Note here, we set $v_2 = 1$ for the sake of simplicity as K. Shimomura did in [13]. We can recover the $v_2$'s by comparing their internal degrees if necessary.

Consider the commutitive diagram of $\Sigma(2)$ comodules

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2^0 & \xrightarrow{1/2} & L_1^1 & \xrightarrow{2} & L_1^1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1/v_1} & & \downarrow{\scriptstyle 1/v_1} & & \downarrow{\scriptstyle 1/v_1} & & \\
0 & \longrightarrow & M_1^1 & \xrightarrow{1/2} & M_0^2 & \xrightarrow{2} & M_0^2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_1} & & \\
0 & \longrightarrow & M_1^1 & \xrightarrow{1/2} & M_0^2 & \xrightarrow{2} & M_0^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We see from $\eta_R(v_1) = v_1$, that this induces the commutative diagram of the cobar complexes $C_{\Sigma(2)}^*M$, and then we get the commutative diagram (1.2) and Lemma 1.4.

## 3 The *Ext* groups $H^*L_1^1[t_1]$

Before starting the computation of the *Ext* modules $H^*L_1^1[t_1]$ by the 2-Bockstein spectral sequence $H^*M_2^0[t_1] \Rightarrow H^*L_1^1[t_1]$, we give some homologous relations in $H^2M_2^0[t_1]$ at the beginning of this section.

**Lemma 3.1.** *In $H^2M_2^0[t_1]$, we have the following homologous relations:*

$$h_{21}^2 = v_2^2 h_{20}^2$$

$$h_{30}^2 = v_2^{-2} h_{31}(h_{21} + v_2 h_{20}) + v_2^{-3} v_3^2 h_{21} h_{20}$$

$$h_{31}^2 = v_2^5 h_{30}(h_{21} + v_2 h_{20}) + v_2^4 v_3 h_{20}(h_{21} + v_2 h_{20}) + v_2^{-2} v_3^4 h_{20}^2.$$

*Proof.* From its definition, we see that $h_{20}^2$, $h_{21}^2$ are represented by $t_2 \otimes t_2$ and $t_2^2 \otimes t_2^2$ respectively. From (2.2) and (2.3), we compute that

$$d(v_2^{-1} t_4) = v_2^{-1} t_2 \otimes t_2^4 + t_2^2 \otimes t_2^2 = v_2^2 t_2 \otimes t_2 + t_2^2 \otimes t_2^2.$$

This proves the first homologous relation.

For $h_{30}^2$, which is represented by $t_3 \otimes t_3$, we compute that mod $(v_2 - 1)$

$$\begin{aligned} d(t_5^2) &= t_3^4 \otimes t_3^4 + t_3^2 \otimes t_2^{16} + t_2^2 \otimes t_3^8 + v_3^2 t_2^8 \otimes t_2^8 \\ &= (t_3 + v_3 t_2^2 + v_3^4 t_2) \otimes (t_3 + v_3 t_2^2 + v_3^4 t_2) + t_3^2 \otimes t_2 \\ &\quad + t_2^2 \otimes (t_3^2 + v_3^2 t_2 + v_3^8 t_2^2) + v_3^2 t_2^2 \otimes t_2^2 \\ &= t_3 \otimes t_3 + \underline{v_3^8 t_2 \otimes t_2}_1 + t_3^2 \otimes t_2 + \underline{t_2^2 \otimes t_3^2}_c \\ &\quad + v_3^2 t_2^2 \otimes t_2 + \underline{v_3^8 t_2^2 \otimes t_2^2}_2 + v_3(t_3 \otimes t_2^2 + t_2^2 \otimes t_3) \\ &\quad + v_3^4(t_3 \otimes t_2 + t_2 \otimes t_3) + v_3^5(t_2^2 \otimes t_2 + t_2 \otimes t_2^2) \end{aligned}$$

$$d(v_3^8 t_4) = \underline{v_3^8 t_2 \otimes t_2}_1 + \underline{v_3^8 t_2^2 \otimes t_2^2}_2$$

$$d(v_3 t_3 \cdot t_2^2) = v_3(t_3 \otimes t_2^2 + t_2^2 \otimes t_3)$$

$$d(v_3^4 t_3 \cdot t_2) = v_3^4(t_3 \otimes t_2 + t_2 \otimes t_3)$$

$$d(v_3^5 t_2^3) = v_3^5(t_2^2 \otimes t_2 + t_2 \otimes t_2^2)$$

$$d(t_3^2 t_2^2) = t_3^2 \otimes t_2^2 + \underline{t_2^2 \otimes t_3^2}_c.$$

Thus, the sum of them gives $u$ such that

$$d(u) = t_3 \otimes t_3 + t_3^2 \otimes (t_2^2 + t_2) + v_3^2 t_2^2 \otimes t_2.$$

By recovering the $v_2$'s, we get the second one.

For the third one, we have

$$\begin{aligned} d(u^2) &= t_3^2 \otimes t_3^2 + t_3^4 \otimes (t_2^2 + t_2) + v_3^4 t_2 \otimes t_2^2 \\ &= t_3^2 \otimes t_3^2 + t_3 \otimes (t_2^2 + t_2) + v_3 t_2^2 \otimes (t_2^2 + t_2) + v_3^4 t_2 \otimes t_2. \end{aligned}$$

By recovering the $v_2$'s, we get the third one from $h_{20}(h_{21} + h_{20}) = h_{21}(h_{21} + h_{20})$. □

Consider the short exact sequence

$$0 \longrightarrow M_2^0[t_1] \xrightarrow{1/2} L_1^1[t_1] \xrightarrow{2} L_1^1[t_1] \longrightarrow 0$$

and the induced long exact sequence

$$\cdots \longrightarrow H^s M_2^0[t_1] \xrightarrow{1/2} H^s L_1^1[t_1] \xrightarrow{2} H^s L_1^1[t_1] \xrightarrow{\delta_1} H^{s+1} M_2^0[t_1] \longrightarrow \cdots.$$

By [8, Remark 3.11], we can determine $H^s L_1^1[t_1]$ to be the 2-torsion module $B^s$, if $B^s$ makes the following sequences exact and commutative

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^s M_2^0[t_1] & \xrightarrow{1/2} & B^s & \xrightarrow{2} & B^s & \xrightarrow{\delta_1} & H^{s+1} M_2^0[t_1] & \longrightarrow \cdots \\ & \downarrow{=} & & \downarrow{i} & & \downarrow{i} & & \downarrow{=} & \\ \cdots \longrightarrow & H^s M_2^0[t_1] & \xrightarrow{1/2} & H^s L_1^1[t_1] & \xrightarrow{2} & H^s L_1^1[t_1] & \xrightarrow{\delta_1} & H^{s+1} M_2^0[t_1] & \longrightarrow \cdots. \end{array}$$

**Lemma 3.2.** *For any $n > 0$, there exists a cochain $R_n \in C^1_{\Sigma(2)} v_2^{-1} BP_*$, which represents the generator $\rho_2$ of $H^1 M_2^0[t_1]$ and*

$$d(R_n) = 0 \quad \bmod(2^n, v_1).$$

*Proof.* Let $R = t_4 + t_4^2 + v_3 t_3^2 + v_3^2 t_3 + v_3^4 t_3^2 + v_3^3(t_2^2 + t_2) + v_3^6 t_2^2$ be the cochain of $C^1_{\Sigma(2)} v_2^{-1} BP_*$ with inner degree 0. Then $R$ represents the generator $\rho_2 \in H^1 M_2^0[t_1]$. From (2.3) we compute that $\bmod(2, v_1)$

$$\begin{aligned} R^2 + R = {} & t_4 + t_4^2 + v_3 t_3^2 + v_3^2 t_3 + v_3^4 t_3^2 + v_3^3(t_2^2 + t_2) + v_3^6 t_2^2 \\ & + t_4^2 + t_4^4 + v_3^2 t_3^4 + v_3^4 t_3^2 + v_3^8 t_3^4 + v_3^6(t_2^4 + t_2^2) + v_3^{12} t_2^4 \end{aligned}$$

$$= v_3(t_3^8 + t_3^2) + v_3^2(t_3^4 + t_3) + v_3^8(t_3^4 + t_3)$$

$$+ v_3^3(t_2^2 + t_2) + v_3^6 t_2 + v_3^{12} t_2$$

$$= 0.$$

Thus $d(R) = 0 \bmod (2, v_1)$ and $d(R^{2^n}) = 0 \bmod (2^n, v_1)$. Denote $R^{2^n}$ by $R_n$, then we get the lemma. □

**Lemma 3.3.** *For the connecting homomorphism* $\delta_1 : H^s L_1^1[t_1] \to H^{s+1} M_2^0[t_1]$ *we have:*

$$\delta_1(v_2/2) = h_{20}$$

$$\delta_1(v_3/2) = h_{30}$$

$$\delta_1(v_2^{2^n}/2^{n+1}) = v_2^{2^n-2}(h_{21} + v_2 h_{20})$$

$$\delta_1(v_3^{2^m}/2^{m+1}) = v_2^{2^m-2}(v_3 h_{30} + h_{31})$$

$$\delta_1(h_{31}/2) = v_2^{-1} h_{31} h_{20} + v_2^{-2} h_{31} h_{21} + v_2^{-3} v_3^2 h_{21} h_{20} = \delta(v_3 h_{30}/2)$$

$$\delta_1(v_2 h_{31}/2) = v_2^{-1} h_{31} h_{21} + v_2^{-2} v_3^2 h_{21} h_{20}$$

$$\delta_1(v_2 v_3 h_{31}/2) = v_2 h_{30} h_{31} + v_2^{-1} v_3 h_{31} h_{21} + v_2^{-2} v_3^3 h_{21} h_{20}$$

$$\delta_1(v_2 v_3 h_{30} h_{31}/2) = v_2^{-1} v_3 h_{30} h_{21} h_{31} + v_2^{-2} v_3^2 (h_{31} + v_3 h_{30}) h_{21} h_{20}$$

$$+ v_2^{-3} v_3^4 h_{20}^2 (h_{21} + v_2 h_{20})$$

$$\delta_1(v_2 v_3 h_{21} h_{31}/2) = v_2 h_{30} h_{21} h_{31} + v_3^3 h_{20}^3.$$

*Proof.* The 1st, 2nd, 3rd and 4th come from direct computation. The 5th comes from Lemma 3.1 and the fact that

$$\delta_1(h_{31}/2) = \delta_1(v_3 t_3/2) = h_{30}^2 = v_2^{-1} h_{31} h_{20} + v_2^{-2} h_{31} h_{21} + v_2^{-3} v_3^2 h_{21} h_{20}.$$

Then the 6th and the 7th come from

$$\delta_1(v_2 h_{31}/2) = \delta_1(v_2/2) h_{31} + v_2 \delta_1(h_{31}/2)$$

$$\delta_1(v_2 v_3 h_{31}/2) = \delta_1(v_3/2) v_2 h_{31} + v_3 \delta_1(v_2 h_{31}/2).$$

For the 8th, we compute that

$$
\begin{aligned}
&\delta_1(v_2v_3h_{30}h_{31}/2)\\
&= \delta_1(v_2v_3h_{31}/2)h_{30}\\
&= v_2h_{30}^2h_{31} + v_2^{-1}v_3h_{30}h_{21}h_{31} + v_2^{-2}v_3^3h_{30}h_{21}h_{20}\\
&= v_2^{-1}h_{31}^2(h_{21} + v_2h_{20}) + v_2^{-2}v_3^2h_{21}h_{20}h_{31} + v_2^{-1}v_3h_{30}h_{21}h_{31} + v_2^{-2}v_3^3h_{30}h_{21}h_{20}\\
&= v_2^4h_{30}(h_{21} + v_2h_{20})^2 + v_2^3v_3h_{20}(h_{21} + v_2h_{20})^2\\
&\quad + v_2^{-3}v_3^4h_{20}^2(h_{21} + v_2h_{20}) + v_2^{-2}v_3^2h_{21}h_{20}h_{31}\\
&\quad + v_2^{-1}v_3h_{30}h_{21}h_{31} + v_2^{-2}v_3^3h_{30}h_{21}h_{20}\\
&= v_2^{-1}v_3h_{30}h_{21}h_{31} + v_2^{-2}v_3^2(h_{31} + v_3h_{30})h_{21}h_{20} + v_2^{-3}v_3^4h_{20}^2(h_{21} + v_2h_{20}).
\end{aligned}
$$

And for the 9th, we compute that

$$
\begin{aligned}
\delta_1(v_2v_3h_{21}h_{31}/2) &= \delta_1(v_2v_3h_{31}/2)h_{21} + v_2v_3h_{31}h_{20}^2\\
&= v_2h_{30}h_{21}h_{31} + v_2^{-1}v_3h_{31}h_{21}^2 + v_2^{-2}v_3^3h_{21}^2h_{20} + v_2v_3h_{31}h_{20}^2\\
&= v_2h_{30}h_{21}h_{31} + v_3^3h_{20}^3. \qquad \square
\end{aligned}
$$

To determine the *Ext* module $H^*L_1^1[t_1]$ by the 2-Bockstein spectral sequence $H^*M_2^0[t_1] \Rightarrow H^*L_1^1[t_1]$, we divide $H^*M_2^0[t_1]$ into the tensor product of $\Lambda(\rho_2)$ and the direct sum of $C_i$:

$$
\begin{aligned}
&C_0 = K_*^2[v_3^2] \otimes \Lambda(h_{21}, v_3h_{30})\\
&C_1 = K_*^2[v_3^2]\{v_3, v_3h_{21}, h_{30}, h_{30}h_{21}\}\\
&C_2 = v_2h_{31}K_*^2[v_3^2] \otimes \Lambda(v_3, h_{21}, h_{30})\\
(3.4)\quad &C_3 = v_2K_*^2[v_3^2, h_{20}] \otimes \Lambda(v_3, h_{21}, h_{30})\\
&\qquad \oplus h_{20}K_*^2[v_3^2, h_{20}] \otimes \Lambda(v_3, h_{21}, h_{30})\\
&C_4 = h_{31}K_*^2[v_3^2, h_{20}] \otimes \Lambda(v_3, h_{21}, h_{30})\\
&\qquad \oplus v_2^{-1}h_{20}h_{31}K_*^2[v_3^2, h_{20}] \otimes \Lambda(v_3, h_{21}, h_{30})
\end{aligned}
$$

where $K_*^2 = Z/2[v_2^{\pm 2}]$, from which we get:

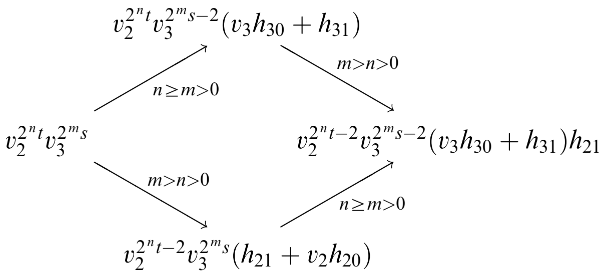
**Theorem 3.5.** *The module* $H^*L_1^1[t_1]$ *is the tensor product of* $\Lambda(\rho_2)$ *and the direct sum of* $\widetilde{C_i}$*:*

$$\begin{aligned}
\widetilde{C_0} &= Z/2^{m+1}\{v_2^{2^n t}v_3^{2^m s}/2^{m+1} \mid \textit{for } s \textit{ odd and } n \geq m > 0\}\\
&\oplus Z/2^{n+1}\{v_2^{2^n t}v_3^{2^m s}/2^{n+1} \mid \textit{for } t \textit{ odd and } m > n > 0\}\\
&\oplus Z/2^{n+1}\{v_2^{2^n t}v_3^{2^m s-2}v_3h_{30}/2^{n+1} \mid m > n > 0\}\\
&\oplus Z/2^{m+1}\{v_2^{2^n t-2}v_3^{2^m s}h_{21}/2^{m+1} \mid n \geq m > 0\}\\
&\oplus \mathbf{Q}/Z_{(2)}\{1, v_2^{-2}h_{21}\}\\
\widetilde{C_1} &= K_*^2[v_3^2]\{v_3/2, v_3h_{21}/2\}\\
\widetilde{C_2} &= v_2h_{31}K_*^2[v_3^2]\{1/2, v_3/2, v_3h_{21}/2, v_3h_{30}/2\}\\
\widetilde{C_3} &= v_2/2K_*^2[v_3^2, h_{20}] \otimes \Lambda(v_3, h_{21}, h_{30})\\
\widetilde{C_4} &= h_{31}/2K_*^2[v_3^2, h_{20}] \otimes \Lambda(v_3, h_{21}, h_{30}).
\end{aligned}$$

*Proof.* For the submodules $C_i$ of $H^*M_2^0[t_1]$ in (3.4) and the submodules $\widetilde{C_i}$ of $H^*L_1^1[t_1]$ in Theorem 3.5, we have the following exact sequences for $0 \leq i \leq 4$

$$\cdots \longrightarrow C_i^s \xrightarrow{1/2} \widetilde{C_i}^s \xrightarrow{2} \widetilde{C_i}^s \xrightarrow{\delta_1} C_i^{s+1} \longrightarrow \cdots.$$

Indeed, from Lemma 3.3, we see that the connecting homomorphism $\delta_1 : H^sL_1^1[t_1] \to H^{s+1}M_2^0[t_1]$ behaves on $\widetilde{C_0}$ in the form of




Then we see the exact sequence for $i = 0$.

From Lemma 3.3, we see that

$$\delta_1 : K_*^2[v_3^2]\{v_3/2, v_3h_{21}/2\} \to K_*^2[v_3^2]\{h_{30}, h_{30}h_{21}\}$$

and

$$\delta_1 : v_2h_{31}K_*^2[v_3^2]\{1/2, v_3/2, v_3h_{30}/2, v_3h_{21}\} \to$$
$$v_2h_{31}K_*^2\{v_2^{-2}h_{21}, h_{30}, v_2^{-2}v_3h_{30}h_{21}, h_{30}h_{21}\}.$$

This proves the exactness for $i = 1$, and 2.

Note that for any generator of $x/2 \in 1/2K_*^2[v_3^2, h_{20}] \otimes \Lambda(v_3, h_{21}, h_{30})$, $\delta_1(x/2)$ does not have a term $xh_{20}$ as a summand. Then from $\delta_1(v_2/2) = h_{20}$ and $\delta_1(h_{31}/2) = v_2^{-1}h_{20}h_{31} + \cdots$ we get the exactness for $i = 3$ and 4.

From Lemma 3.2, we see that $\delta_1(\rho_2/2^n) = 0$ for any $n$. This indices the following exact sequences for $0 \leq i \leq 4$

$$\cdots \longrightarrow C_i^s \otimes \Lambda(\rho_2) \xrightarrow{1/2} \widetilde{C_i}^s \otimes \Lambda(\rho_2) \xrightarrow{2} \widetilde{C_i}^s \otimes \Lambda(\rho_2) \xrightarrow{\delta_1} C_i^{s+1} \otimes \Lambda(\rho_2) \longrightarrow \cdots.$$

Then from $H^*M_2^0[t_1] = (\bigoplus_{i=0}^4 C_i) \otimes \Lambda(\rho_2)$ we get the theorem. □

## 4 The homotopy groups $\pi_*(L_2T(1)/(v_1))$

Let $T(1)/(v_1)$ be the cofiber of the map $v_1 : \Sigma^2T(1) \to T(1)$ and $T(1)/(2^\infty, v_1)$ be the cofiber of the localization $T(1)/(v_1) \to L_0T(1)/(v_1)$. Then we have the map $2 : T(1)/(v_1) \to T(1)/(v_1)$ which makes that the following diagram commutes:

$$\begin{array}{ccccc} T(1)/(v_1) & \longrightarrow & L_0T(1)/(v_1) & \longrightarrow & T(1)/(2^\infty, v_1) \\ \downarrow 2 & & \downarrow 2 & & \downarrow 2 \\ T(1)/(v_1) & \longrightarrow & L_0T(1)/(v_1) & \longrightarrow & T(1)/(2^\infty, v_1). \end{array}$$

The $3 \times 3$ Lemma shows that the fiber of $2 : T(1)/(2^\infty, v_1) \to T(1)/(2^\infty, v_1)$ is the cofiber $T(1)/(2, v_1)$ of $2 : T(1)/(v_1) \to T(1)/(v_1)$. Therefore we have a cofiber sequence

$$L_2T(1)/(2, v_1) \xrightarrow{\varphi_1} L_2T(1)/(2^\infty, v_1) \xrightarrow{2} L_2T(1)/(2^\infty, v_1)$$

which induces the short exact sequence of $BP_*$-homology

$$0 \longrightarrow M_2^0[t_1] \xrightarrow{1/2} L_1^1[t_1] \xrightarrow{2} L_1^1[t_1] \longrightarrow 0.$$

In [6], it was shown that the $E_2$-term of the Adams-Novikov spectral sequence for $L_2T(1)/(2, v_1)$ is

$$E_2^s = H^sM_2^0[t_1] = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2)$$

and the Adams-Novikov differentials were shown to be

$$(4.1)\qquad d_r(v_3)=0 \quad d_r(v_3^4)=0 \qquad d_r(h_{20})=0 \quad d_r(h_{21})=0$$

$$d_r(h_{30})=0 \quad d_3(h_{31})=v_2h_{20}^4 \quad d_r(\rho_2)=0$$

as a module over $K(2)_*[v_3^4, h_{20}]\otimes\Lambda(h_{21}, h_{31})$ and all the other differentials were deduced by $d_3(x\cdot y)=\delta(x)\delta(y)h_{20}$ for permanent cycles $x$ and $y$, where $\delta: H^sM_2^0[t_1] \to H^{s+1}M_2^0[t_1]$ is the connecting homomorphism induced by

$$0\to M_2^0[t_1]\to v_2^{-1}BP_*[t_1]/(2, v_1^2)\to M_2^0[t_1]\to 0.$$

Then, by a suitable replacement, the Adams-Novikov differentials are

$$d_3(h_{31})=v_2h_{20}^4 \qquad d_r(v_3)=0 \quad d_3(v_3^2)=v_2^2h_{20}^3$$

$$d_3(v_3^3)=v_2^2v_3h_{20}^3 \qquad d_r(h_{30})=0 \quad d_r(v_3h_{30})=0$$

$$d_3(v_3^2h_{30})=v_2^2h_{30}h_{20}^3 \quad d_3(v_3^3h_{30})=v_2^2v_3h_{30}h_{20}^3+v_3^2h_{21}h_{20}^3$$

as a module over $K(2)_*[v_3^4, h_{20}]\otimes\Lambda(h_{21}, h_{31}, \rho_2)$.

**Lemma 4.2.** *For the elements $x$ in the submodule $\widetilde{C_3}$ and $\widetilde{C_4}$, we have $d_r(x)=0$ except for:*

$$d_3(v_3^{4t+2}\alpha/2)=v_2^2v_3^{4t}\alpha h_{20}^3/2$$

$$d_3(v_3^{4t+3}\alpha/2)=v_2^2v_3^{4t+1}\alpha h_{20}^3/2.$$

*Proof.* Note that all the elements in $\widetilde{C_3}$ and $\widetilde{C_4}$ are killed by 2, and so the Lemma is obtained from the image of $\varphi_1: L_2T(1)/(2, v_1)\to L_2T(1)/(2^\infty, v_1)$. □

**Lemma 4.3.** *For the elements $x\in\widetilde{C_0}$, the 3rd Adams-Novikov differentials $d_3(x)=0$ except for:*

$$d_3(v_2^{2^nt}v_3^{2^ms}/2^{m+1})=v_2^{2^nt+3}v_3^{2^ms-4}v_3h_{30}h_{20}^2/2 \qquad \textit{for } n\geq m>1$$

$$d_3(v_2^{2^nt}v_3^{2^m-2}v_3h_{30}/2^{n+1})=v_2^{2^nt+1}v_3^{2^ms-4}v_3h_{30}h_{21}h_{20}^2/2 \quad \textit{for } m>n>0$$

$$d_3(v_2^{2^nt-2}v_3^{2^ms}h_{21}/2^{m+1})=v_2^{2^nt+1}v_3^{2^ms-4}v_3h_{30}h_{21}h_{20}^2/2 \quad \textit{for } n\geq m>1.$$

*Proof.* From the proof of the Theorem 3.5, we see that the connecting homomorphism $\delta_1: H^sL_1^1[t_1]\to H^{s+1}M_2^0[t_1]$ is an injection for $s>2$. By the naturality of the differentials, we compute that for $m>1$

$$\delta_1(d_3(v_2^{2^n t}v_3^{2^m s}/2^{m+1}))$$

$$= d_3(\delta_1(v_2^{2^n t}v_3^{2^m s}/2^{m+1}))$$

$$= d_3(v_2^{2^n t}v_3^{2^m s-2}(v_3h_{30}+h_{31}))$$

$$= v_2^{2^n t+2}v_3^{2^m s-4}(v_3h_{30}+h_{31})h_{20}^3 + v_2^{2^n t}v_3^{2^m s-2}(h_{21}+v_2h_{20})h_{20}^3.$$

And from

$$\delta_1(v_2^{2^n t+3}v_3^{2^m s-4}((v_3h_{30}+h_{31})+v_2^{-2}v_3^2(h_{21}+v_2h_{20}))h_{20}^2/2)$$

$$= v_2^{2^n t+2}v_3^{2^m s-4}((v_3h_{30}+h_{31})+v_2^{-2}v_3^2(h_{21}+v_2h_{20}))h_{20}^3,$$

we get the first Adams-Novikov differential.

Note that, both $v_2^{2^n t}v_3^{2^m s-2}(v_3h_{30}+h_{31})/2^{n+1}$ for $m > n > 0$ and $v_2^{2^n t-2}v_3^{2^m s}(h_{21}+v_2h_{20})/2^{m+1}$ for $n \geq m > 1$ are sent to

$$v_2^{2^n t-2}v_3^{2^m s-2}(v_3h_{30}+h_{31})(h_{21}+v_2h_{20})$$

by the connecting homomorphism $\delta_1 : H^1L_1^1[t_1] \to H^2M_2^0[t_1]$. From

$$d_3(v_2^{2^n t-2}v_3^{2^m s-2}(v_3h_{30}+h_{31})(h_{21}+v_2h_{20}))$$

$$= v_2^{2^n t}v_3^{2^m s-4}(v_3h_{30}+h_{31})(h_{21}+v_2h_{20})h_{20}^3$$

we get the second and the third Adams-Novikov differentials from

$$\delta_1(v_2(v_3h_{30}+h_{31})(h_{21}+v_2h_{20})h_{20}^2/2) = (v_3h_{30}+h_{31})(h_{21}+v_2h_{20})h_{20}^3.$$

From $\delta_1(v_2^{2^n t}v_3^{2^m s}/2^{n+1}) = v_2^{2^n t-2}v_3^{2^m s}(h_{21}+v_2h_{20})$ for $m > n > 0$ and

$$d_3(v_2^{2^n t-2}v_3^{2^m s}(h_{21}+v_2h_{20})) = 0$$

we see that

$$d_3(v_2^{2^n t}v_3^{2^m s}/2^{n+1}) = 0. \qquad \square$$

To compute the Adams-Novikov differentials on the generators in $\widetilde{C_1}$ and $\widetilde{C_2}$, we need some homologous relations in $H^*L_1^1[t_1]$.

**Lemma 4.4.** *In $H^*L_1^1[t_1]$, we have the following homologous relations:*

$$v_3h_{20}/2 = v_2h_{30}/2$$

$$v_3h_{21}h_{20}/2 = (v_2h_{30}h_{21} + v_2v_3h_{20}^2)/2$$

$$v_2v_3h_{21}h_{31}h_{20}/2 = (v_2^2h_{30}h_{21}h_{31} + v_2v_3^3h_{20}^3)/2$$

$$v_2h_{31}h_{20}/2 = (h_{21}h_{31} + v_2^{-1}v_3^2h_{21}h_{20})/2$$

$$v_2v_3h_{31}h_{20}/2 = (v_3h_{21}h_{31} + v_2^2h_{30}h_{31} + v_2^{-1}v_3^2h_{21}h_{20})/2$$

$$v_2v_3h_{30}h_{31}h_{20}/2 = (v_3h_{30}h_{21}h_{31} + \cdots)/2.$$

*Proof.* Consider the connecting homomorphism $\delta_1 : H^*L_1^1[t_1] \to H^*M_2^0[t_1]$, we see that an element $x/2 = 0 \in H^{s+1}L_1^1[t_1]$ if $\delta_1(y/2) = x$. Now we have

$$\delta_1(v_2v_3/2) = v_3h_{20} + v_2h_{30}.$$

Thus $(v_3h_{20} + v_2h_{30})/2 = 0$, and this implies $v_3h_{20}/2 = v_2h_{30}/2$. Similarly, from

$$\delta_1(v_2^2h_{31}/2) = v_2h_{31}h_{20} + h_{31}h_{21} + v_2^{-1}v_3^2h_{21}h_{20}$$

we see that

$$v_2h_{31}h_{20}/2 = (h_{31}h_{21} + v_2^{-1}v_3^2h_{21}h_{20})/2.$$

In the same way, we get all the others from the proof of Theorem 3.5. □

**Lemma 4.5.** *For the elements $x$ in $\widetilde{C_1}$ and $\widetilde{C_2}$, we have $d_r(x) = 0$ for all $r > 0$ except for:*

$$d_3(v_3^3/2) = v_2^3h_{30}h_{20}^2/2$$

$$d_3(v_3^3h_{21}/2) = v_2^3h_{30}h_{21}h_{20}^2/2 \qquad \textit{in } \widetilde{C_1}$$

$$d_3(v_2v_3^3h_{21}h_{31}/2) = v_2^4h_{30}h_{21}h_{31}h_{20}^2/2 + v_2^3v_3^2h_{30}h_{21}h_{20}^3/2$$

$$d_3(v_2v_3^2h_{31}/2) = v_2^2h_{21}h_{31}h_{20}^2/2$$

$$d_3(v_2v_3^3h_{31}/2) = v_2^2v_3h_{21}h_{31}h_{20}^2/2$$

$$d_3(v_2v_3^3h_{30}h_{31}/2) = v_2^2v_3h_{30}h_{21}h_{31}h_{20}^2/2 \qquad \textit{in } \widetilde{C_2}.$$

*Proof.* Note that the elements in $\widetilde{C_1}$ and $\widetilde{C_2}$ are killed by 2 also. Then from the map $\varphi_1 : L_2T(1)/(2, v_1) \to L_2T(1)/(2^\infty, v_1)$ and (4.1) we see that:

$$d_3(v_3^3/2) = v_2^2v_3h_{20}^3/2$$

$$d_3(v_3^3h_{21}/2) = v_2^2v_3h_{21}h_{20}^3/2$$

$$d_3(v_2v_3^3h_{21}h_{31}/2) = v_2^3v_3h_{21}h_{31}h_{20}^3/2 + v_2^2v_3^3h_{21}h_{20}^4$$

$$d_3(v_2v_3^2h_{31}/2) = v_2^3h_{31}h_{20}^3/2 + \cdots$$

$$d_3(v_2v_3^3h_{31}/2) = v_2^3v_3h_{31}h_{20}^3/2 + \cdots$$

$$d_3(v_2v_3^3h_{30}h_{31}/2) = v_2^3v_3h_{30}h_{31}h_{20}^3/2 + \cdots.$$

In the last three Adams-Novikov differentials, a detailed computation shows that the $\cdots$ parts do not have $v_2h_{21}h_{20}^2h_{31}\alpha/2$ as a summand for $\alpha = v_2, v_2v_3$ and $v_2v_3h_{30}$ respectively. Thus from the homologous relations given in Lemma 4.4, we get the lemma. □

**Proposition 4.6.** *The homotopy group $\pi_*(L_2T(1)/(2^\infty, v_1))$ is the tensor product of $\Lambda(\rho_2)$ and the direct sum of $\widehat{C_i}$, where*

$$\widehat{C_0} = Z/2^m\{v_2^{2^nt}v_3^{2^ms}/2^m \mid n \geq m > 1\}$$

$$\oplus Z/2^{n+1}\{v_2^{2^nt}v_3^{2^ms}/2^{n+1} \mid m > n > 0\}$$

$$\oplus Z/4\{v_2^{2t}v_3^{4s+2}/4 \mid t \in Z\}$$

$$\oplus \mathbf{Q}/Z_{(2)}\{1, v_2^{-2}h_{21}\}$$

$$\widehat{C_1} = K_*^2[v_3^4]\{v_3, v_3h_{21}\}$$

$$\widehat{C_2} = v_2h_{31}K_*^2[v_3^4]\{1, v_3, v_3h_{21}, v_3h_{30}\}$$

$$\widehat{C_3} = v_2K_*^2[v_3^4, h_{20}]/(h_{20}^3) \otimes \Lambda(v_3, h_{21})$$

$$\oplus v_2K_*^2[v_3^4, h_{20}]/(h_{20}^2)\{h_{30}, h_{30}h_{21}, v_3h_{30}h_{21}\}$$

$$\oplus K_*^2[h_{20}]/(h_{20}^2)\{v_2^{2^nt+3}v_3^{2^ms-4}v_3h_{30} \mid n \geq m > 1\}$$

$$\oplus K_*^2[h_{20}]/(h_{20}^3)\{v_2^{2^nt+3}v_3^{2^ms-4}v_3h_{30} \mid m > n > 0 \text{ or } m = 1\}$$

$$\widehat{C_4} = h_{31}K_*^2[v_3^4, h_{20}]/(h_{20}^3) \otimes \Lambda(v_3, h_{30})$$

$$\oplus h_{31}h_{21}K_*^2[v_3^4, h_{20}]/(h_{20}^2) \otimes \Lambda(v_3, h_{30}).$$

*Proof.* From the above computations, we see that the $E_4$-term $E_4^{s,*}$ for $\pi_*(L_2T(1)/(2^\infty, v_1))$ is zero for $s > 5$. Then we get the proposition, if all of the higher Adams-Novikov differentials $d_r = 0$ for $r > 3$.

Suppose $d_5(v_2^{2^n t}v_3^{2^m s}/2^l) = h_{30}h_{20}\rho_2 h_{31}y/2$ for some $y = v_2^{2a}v_3^{4b}h_{21}$ or $v_2^{2a}v_3^{4b+1}h_{30}$ (Note that $v_2^{2a}v_3^{4b}h_{30}h_{20}^2\rho_2 h_{31}/2$ and $v_2^{2a}v_3^{4b+1}h_{30}h_{21}h_{20}\rho_2 h_{31}/2$ have different inner degrees, and then these two generators are the only possibilities). Now map this to $L_2T(1)/(2, v_1)$, we have a contradiction

$$0 = d_5(0) = v_2^{2a-2}v_3^{4b}h_{30}h_{20}\rho_2 h_{31}(h_{21} + v_2 h_{20})y + \cdots. \qquad \square$$

**Theorem 4.7.** *The homotopy groups of $L_2T(1)/(v_1)$ are isomorphic to*

$$\pi_*(L_2T(1)/(v_1)) = Z_{(2)} \oplus (\pi_*(L_2T(1)/(2^\infty, v_1)) - \mathbf{Q}/Z_{(2)}).$$

*Proof.* From $H^*2^{-1}BP_*[t_1]/(v_1) = \mathbf{Q}$, we see that $\pi_*(L_0T(1)/(v_1)) = \mathbf{Q}$ concentrated in degree 0. Consider the long exact exact sequence of homotopy groups

$$\cdots \to \pi_t(L_2T(1)/(v_1)) \to \pi_t(L_0T(1)/(v_1)) \to \pi_t(L_2T(1)/(2^\infty, v_1)) \to \cdots$$

induced by the cofiber sequence

$$L_2T(1)/(v_1) \to L_0T(1)/(v_1) \to L_2T(1)/(2^\infty, v_1),$$

then we see the theorem. $\square$

## 5 The homomorphism $\Delta_1$ and its application

Consider the maps $\Delta_1 : H^sL_1^1[t_1] \to H^{s+1}M_1^1[t_1]$ and $\Delta_2 : H^sM_1^1[t_1] \to H^{s+1}L_1^1[t_1]$ given in (1.3). Then from the commutativity of diagram (1.2), we see that $\beta/v_1 \in im\delta_1 : H^*M_0^2[t_1] \to H^*M_1^1[t_1]$ if $\Delta_1(\alpha/2^l) = \beta/v_1$.

**Lemma 5.1.** *From the connecting homomorphisms $\delta_1 : H^sL_1^1[t_1] \to H^{s+1}M_2^0[t_1]$ given in Section 3, we compute that:*

$$\Delta_1(v_2^{2^n t}v_3^{2^m s}/2^{m+1}) = v_2^{2^n t}v_3^{2^m s-2}v_3 h_{30}/v_1 \qquad \textit{for } n \geq m > 0$$

$$\Delta_1(v_2^{2^n t}v_3^{2^m s}/2^{n+1}) = 0 \qquad \textit{for } m > n > 0$$

$$\Delta_1(v_2^{2^n t}v_3^{2^m s-2}v_3 h_{30}/2^{n+1}) = v_2^{2^n t-2}v_3^{2^m s-2}v_3 h_{30}h_{21}/v_1 \qquad \textit{for } m > n > 0$$

$$\Delta_1(v_2^{2^n t-2}v_3^{2^m s}h_{21}/2^{m+1}) = v_2^{2^n t-2}v_3^{2^m s-2}v_3 h_{30}h_{21}/v_1 \qquad \textit{for } n \geq m > 0.$$

*Proof.* Note that

$$\Delta_1(v_2^{2^n t}v_3^{2^m s}/2^{n+1}) = v_2^{2^n t-2}v_3^{2^m s}(h_{21} + v_2 h_{20})/v_1 \quad \text{for } m > n > 0.$$

and from the fact that $v_2^{2^n t-2}v_3^{2^m s}(h_{21}+v_2h_{20})/v_1$ is homologous to 0 in $H^1M_1^1[t_1]$, we see the second. The others come from direct computation.

**Lemma 5.2.** *For $i>0$, we list some of the elements in $\widetilde{C_i}$ that have non-zero images and some of the others that have zero iamge under $\Delta_1$*

$$\Delta_1(v_3/2) = h_{30}/v_1 = (v_2^{-2}v_3h_{21} + c_{30} + v_2^{-8}v_3^4h_{20})/v_1$$

$$\Delta_1(v_2v_3h_{30}h_{31}/2) = v_2^{-1}v_3h_{30}h_{21}h_{31}/v_1$$

$$\Delta_1(v_2v_3^{4t+2}/2) = v_3^{4t+2}h_{20}/v_1$$

$$\Delta_1(v_2v_3^{4^{n+1}t+2\cdot 4^n+4e_n+2}h_{30}/2) = v_3^{4^{n+1}t+2\cdot 4^n+4e_n+2}h_{20}h_{30}/v_1$$

$$\Delta_1(v_3^{2\cdot 4^{n+1}t+4^{n+1}+8e_n+2}h_{31}/2) = v_1^{-1}v_3^{2\cdot 4^{n+1}t+4^{n+1}+8e_n+2}h_{20}h_{31}/v_1$$

$$\Delta_1(v_2v_3^{4s+3}h_{31}/2) = v_2v_3^{4t+2}h_{30}h_{31}/v_1$$

$$\Delta_1(v_2v_3^{4s+1}h_{31}/2) = 0$$

$$\Delta_1(v_2v_3h_{21}h_{31}/2) = 0$$

*where* $c_{30} = h_{30} + v_2^{-2}v_3^2h_{21} + v_2^{-8}v_3^4h_{20}$ *that represents* $\tilde{g}_1 \in H^1M_2^0[t_1]$ *and* $e_n = (4^n-1)/3$.

*For* $\alpha \in K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30})$

$$\Delta_1(v_2v_3\alpha/2) = v_3h_{20}\alpha/v_1 \quad \Delta_1(v_3h_{31}\alpha/2) = v_2^{-1}v_3h_{20}h_{31}\alpha/v_1.$$

*Proof.* It is easy to get all of them from $\delta_1 : H^*L_1^1[t_1] \to H^*M_2^0[t_1]$ except for the two formulae that have zero image under $\Delta_1$.

From Lemma 3.3, we see that

$$\Delta_1(v_2v_3^{4s+1}h_{31}/2 + v_3^{4s+3}h_{20}/2)$$

$$= v_3^{4s}(v_2h_{30}h_{31} + v_2^{-1}v_3^1h_{31}h_{21} + v_2^{-2}v_3^3h_{21}h_{20} + v_3^2h_{30}h_{20})/v_1.$$

Notice that in [13, 6.5], the connecting homomorphism $\delta_2 : H^*M_1^1[t_1] \to H^*M_2^0[t_1]$ is given as

$$\delta_2(v_3^{4^{n+1}t+3\cdot 4^n+4e_{n-1}}c_{30}/v_1^{a_{2n+1}}) = v_3^{2\cdot 4^n(2t+1)-4}c_{30}\bar{c}_{31} \quad k>0$$

$$\delta_2(v_3^{2\cdot 4^{n+1}t+4^{n+1}+8e_n}\bar{c}_{31}/v_1^{a_{2n+2}}) = v_3^{4^{n+1}(2t+1)-4}c_{30}\bar{c}_{31}$$

where $c_{30} = h_{30} + v_2^{-2}v_3h_{21} + v_2^{-8}v_3^4h_{20}$ and $\bar{c}_{31} = h_{31} + v_2^{-1}v_3^2h_{20}$. We compute that $\bmod(v_2 - 1)$

$$\begin{aligned} c_{30}\bar{c}_{31} &= (h_{30} + v_3h_{21} + v_3^4h_{20})(h_{31} + v_3^2h_{20}) \\ &= h_{30}h_{31} + v_3h_{31}h_{21} + v_3^3h_{21}h_{20} + v_3^2h_{30}h_{20} + v_3^4(h_{31} + v_3^2h_{20})h_{20}. \end{aligned}$$

Thus, we have

$$\Delta_1(v_2v_3^{4s+1}h_{31}/2 + v_3^{4s+3}h_{20}/2) = v_3^{4s}c_{30}\bar{c}_{31}/v_1 + v_3^{4s+4}\bar{c}_{31}h_{20}/v_1 = 0$$

and

$$\Delta_1(v_3^{2s}(v_2v_3h_{31} + v_3^3h_{20})(h_{21} + v_2h_{20})/2)$$

$$= v_3^{2s}(c_{30}\bar{c}_{31} + v_3^4\bar{c}_{31}h_{20})(h_{21} + v_2h_{20})/v_1 = 0$$

in $H^2M_1^1[t_1]$. □

From the computation given as above, we get the submodules $\widetilde{A_{20}} \otimes \Lambda(\rho_2)$ of $H^*L_1^1[t_1]$ given in (1.5) and

$$\begin{aligned} (5.3) \qquad \widetilde{A_{21}} &= K_*^2[v_3^2]\{v_3/2, v_2v_3h_{30}h_{31}/2\} \otimes \Lambda(\rho_2) \\ &\oplus K_*^2[v_3^4]\{v_2v_3^2/2, v_2v_3^3h_{31}/2\} \otimes \Lambda(\rho_2) \\ &\oplus Z/2\{v_2v_3^{4t'+2}h_{30}/2 \mid 4t' = (4t+2)4^n + 4e_n, n > 0\} \otimes \Lambda(\rho_2) \\ &\oplus Z/2\{v_2v_3^{8t'+2}h_{31}/2 \mid 8t' = (2t+1)4^{n+1} + 8e_n, n \geq 0\} \otimes \Lambda(\rho_2) \\ &\oplus Z/2^{m+1}\{v_2^{2^nt}v_3^{2^ms}/2^{m+1}, v_2^{2^nt-2}v_3^{2^ms}h_{21}/2^{m+1} \mid n \geq m > 0\} \\ &\oplus Z/2^{n+1}\{v_2^{2^nt}v_3^{2^ms-2}v_3h_{30}/2^{n+1} \mid m > n > 0\}. \end{aligned}$$

**Note:** We just know that there is the generator $v_2^{2^nt-2}v_3^{2^ms}h_{21}/2^{m+1}v_1 \in H^1M_0^2[t_1]$, but we do not know its order, for $v_2^{2^nt-2}v_3^{2^ms}h_{21}/v_1$ not being a generator of $H^1M_1^1[t_1]$.

*Proof of Theorem 1.6.* From [13], we see that for $s > 3$, $H^sM_1^1[t_1]$ is the direct sum of

$$(v_3h_{20}/v_1K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, \rho_2))^s \oplus (v_2v_3h_{31}/v_1K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, \rho_2))^s$$

and

$$(v_2v_3h_{20}/v_1K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, \rho_2))^s \oplus (v_3h_{31}/v_1K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, \rho_2))^s.$$

Notice that $\Delta_1(v_2v_3h_{30}h_{31}\rho_2/2) = v_2^{-1}v_3h_{30}h_{21}h_{31}\rho_2/v_1$. Then from the Lemma 5.1 and 5.2, we see that the submodule

$$(v_3h_{20}/v_1K_*^2[v_3^2,h_{20}] \otimes \Lambda(h_{21},h_{30},\rho_2))^s \oplus (v_2v_3h_{31}/v_1K_*^2[v_3^2,h_{20}] \otimes \Lambda(h_{21},h_{30},\rho_2))^s$$

is contained in the images of $\delta_1 : H^{s-1}M_0^2[t_1] \to H^sM_1^1[t_1]$. For $s > 3$, $(\widetilde{A_{20}} \otimes \Lambda(\rho_2))^s$ is isomorphic to

$$(v_2v_3h_{20}/2v_1K_*^2[v_3^2,h_{20}] \otimes \Lambda(h_{21},h_{30},\rho_2))^s \oplus (v_3h_{31}/2v_1K_*^2[v_3^2,h_{20}] \otimes \Lambda(h_{21},h_{30},\rho_2))^s.$$

Thus we have the following exact sequence for $s > 3$

$$\cdots \to H^sM_1^1[t_1] \to (\widetilde{A_{20}} \otimes \Lambda(\rho_2))^s \to (\widetilde{A_{20}} \otimes \Lambda(\rho_2))^s \xrightarrow{\delta_1} H^{s+1}M_1^1[t_1] \to \cdots$$

From [8, Remark 3.11], we see that $(\widetilde{A_{20}} \otimes \Lambda(\rho_2))^s = H^sM_0^2[t_1]$ for $s > 3$, and then Theorem 1.6 follows. □

Let $T(1)/(2^\infty)$ be the cofiber of the localization map $T(1) \to L_0T(1)$ and $T(1)/(2^\infty, v_1^\infty)$ be the cofiber of the localization map $T(1)/(2^\infty) \to L_1T(1)/(2^\infty)$. From Lemma 2.1, we have the map $v_1 : \Sigma^2T(1)/(2^\infty) \to T(1)/(2^\infty)$ which induces $v_1$ for the $BP_*$-homology. Consider the following commutative diagram

$$\begin{array}{ccccc}
\Sigma^2T(1)/(2^\infty) & \longrightarrow & L_1\Sigma^2T(1)/(2^\infty) & \longrightarrow & \Sigma^2T(1)/(2^\infty, v_1^\infty) \\
\downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_1} \\
T(1)/(2^\infty) & \longrightarrow & L_1T(1)/(2^\infty) & \longrightarrow & T(1)/(2^\infty, v_1^\infty).
\end{array}$$

Then the $3 \times 3$ Lemma shows that the fiber of $v_1 : \Sigma^2T(1)/(2^\infty, v_1^\infty) \to T(1)/(2^\infty, v_1^\infty)$ is the cofiber $T(1)/(2^\infty, v_1)$ of

$$v_1 : \Sigma^2T(1)/(2^\infty) \to T(1)/(2^\infty).$$

Therefore we have a cofiber sequence

$$(5.4) \qquad L_2T(1)/(2^\infty, v_1) \xrightarrow{\varphi_2} L_2\Sigma^2T(1)/(2^\infty, v_1^\infty) \xrightarrow{v_1} L_2T(1)/(2^\infty, v_1^\infty)$$

which indices the short exact sequence

$$0 \longrightarrow L_1^1[t_1] \xrightarrow{1/v_1} M_2^0[t_1] \longrightarrow M_2^0[t_1] \longrightarrow 0.$$

Then from the naturality of the Adams-Novikov spectral sequences, we see that for $\alpha \in K_*^2[v_3^4, h_{20}] \otimes \Lambda(h_{21}, h_{30}, \rho_2)$

$$(5.5)\qquad d_3(v_2v_3^3\alpha/2v_1) = v_2^3v_3\alpha h_{20}^3/2v_1 \qquad d_r(v_2v_3\alpha/2v_1) = 0$$

$$d_3(v_3^3h_{31}\alpha/2v_1) = v_2^2v_3h_{31}\alpha h_{20}^3/2v_1 \quad d_r(v_3h_{31}\alpha/2v_1) = 0.$$

Note here the elements $h_{30}$, $h_{31}$ and $\rho_2$ are replaced.

**Lemma 5.6.** *For the generators in* $\widetilde{A_{21}}$ *we have:*

$$d_3(v_2^{2^nt}v_3^{2^ms}/2^{m+1}v_1) = v_2^{2^nt+3}v_3^{2^ms-4}v_3h_{30}h_{20}^2/2v_1 \qquad n \geq m > 1$$

$$d_3(v_2^{2^nt}v_3^{2^m-2}v_3h_{30}/2^{n+1}v_1) = v_2^{2^nt+1}v_3^{2^ms-4}v_3h_{30}h_{21}h_{20}^2/2v_1 \quad m > n > 0$$

$$d_3(v_2^{2^nt-2}v_3^{2^ms}h_{21}/2^{m+1}v_1) = v_2^{2^nt+1}v_3^{2^ms-4}v_3h_{30}h_{21}h_{20}^2/2v_1 \quad n \geq m > 1$$

$$d_r(v_2v_3h_{30}h_{31}/2v_1) = 0$$

$$d_3(v_2v_3^3h_{30}h_{31}/2v_1) = v_2^2v_3h_{30}h_{21}h_{31}h_{20}^2/2v_1$$

$$d_3(v_2v_3^3h_{31}/2v_1) = v_2^2v_3h_{21}h_{31}h_{20}^2/2v_1$$

$$d_r(v_3/2v_1) = 0$$

$$d_3(v_3^3/2v_1) = v_2^3h_{30}h_{20}^2/2v_1 = v_2v_3h_{21}h_{20}^2/2v_1$$

$$d_3(v_2v_3^2/2v_1) = 0$$

$$d_3(v_2v_3^2h_{30}/2v_1) = 0$$

$$d_3(v_2v_3^2h_{31}/2v_1) = 0.$$

*Proof.* All of the Adams-Novikov differentials come from Lemma 4.2, 4.3, 4.5 and the map $\varphi_2 : L_2\Sigma^2T(1)/(2^\infty, v_1) \to L_2T(1)/(2^\infty, v_1^\infty)$ except for the last four.

In $C^*_{\Sigma(2)}M_0^2$, we have

$$d(v_2v_3t_3 \otimes t_2^2/2v_1^2) = v_2t_3 \otimes t_2^2 \otimes t_2^2/2v_1 + v_2v_3t_2 \otimes t_2 \otimes t_2^2/2v_1.$$

Thus $v_2h_{30}h_{21}^2/2v_1 = v_2^3h_{30}h_{20}^2/2v_1 = v_2v_3h_{21}h_{20}^2/2v_1$, then from Lemma 4.5,

$$d_3(v_3^3/2v_1) = v_2^3h_{30}h_{20}^2/2v_1 = v_2v_3h_{21}h_{20}^2/2v_1.$$

From $d(v_2v_3t_2^2 \otimes t_2/2v_1^2) = v_2t_2^2 \otimes t_2^2 \otimes t_2/2v_1$, we see that $v_2h_{21}^2h_{20}/2v_1 = v_2^3h_{20}^3/2v_1 = 0$. Thus

$$d_3(v_2v_3^2/2v_1) = v_2^3h_{20}^3/2v_1 = 0.$$

From $d(v_2v_3t_3 \otimes t_2^2 \otimes t_2/2v_1^2) = v_2t_2^2 \otimes t_3 \otimes t_2^2 \otimes t_2/2v_1 + v_2v_3t_2 \otimes t_2 \otimes t_2^2 \otimes t_2/2v_1$ we see that $v_2^3h_{30}h_{20}^3/2v_1 + v_2v_3h_{21}h_{20}^3/2v_1 = 0$. Then from (4.1), we have

$$d_3(v_2v_3^2h_{30}/2v_1 + v_2^{-1}v_3^3h_{21}/2v_1) = v_2^3h_{30}h_{20}^3/2v_1 + v_2v_3h_{21}h_{20}^3/2v_1 = 0.$$

Replace $v_2v_3^2h_{30}/2v_1$ by $v_2v_3^2h_{30}/2v_1 + v_2^{-1}v_3^3h_{21}/2v_1$ and then we see that its Adams-Novikov differential is zero.

Similarly we see that

$$d_3(v_2v_3^2h_{31}/2v_1) = v_2^3h_{31}h_{20}^3/2v_1 + v_2^2v_3^2h_{20}^4/2v_1 = 0. \qquad \square$$

*Proof of Theorem 1.7.* Notice that

$$d_3(v_2v_3^3h_{30}h_{31}\rho_2/2v_1) = v_2^2v_3h_{30}h_{21}h_{31}\rho_2h_{20}^2/2v_1.$$

Then from (5.5), we see that the $E_4$-term $E_4^{s,*}$ of the Adams-Novikov spectral sequence for computing $\pi_*(L_2T(1)/(2^\infty, v_1^\infty))$ is zero for $s > 5$. The theorem follows. $\square$

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