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A Four-filtrated May Spectral Sequence and Its Applications

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Abstract In this paper, we introduce a four-filtrated version of the May spectral sequence (MSS), from which we study the general properties of the spectral sequence and give a collapse theorem. We also give an efficient method to detect generators of May E_1-term $E_1^{s,t,b,*}$ for a given $(s,t,b,*)$. As an application, we give a method to prove the non-triviality of some compositions of the known homotopy elements in the classical Adams spectral sequence (ASS).

Keywords stable homotopy groups of spheres, Adams spectral sequence, May spectral sequence

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1 Introduction

To determine the stable homotopy groups of spheres $\pi_*(S^0)$ is one of the important problems in homotopy theory. By now, several methods have been found to reach it. For example we have the classical Adams spectral sequence (ASS) (cf. [1]) based on the Eilenberg–MacLane spectrum KZ/p, whose E_2-term is $\mathrm{Ext}_{A_*}^{s,t}(Z/p, Z/p)$ where A_* is the dual Steenrod algebra, and the Adams differential is given by

$$\widetilde{d}_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}. \tag{1.1}$$

We also have the Adams–Novikov spectral sequence (ANSS) (cf. [1–2]) based on the Brown-Peterson spectrum BP.

The most successful method for computing the Ext groups $\mathrm{Ext}_{A_*}^{s,t}(Z/p, Z/p)$ is the May spectral sequence (MSS) (cf. [3]). Unfortunately, this material has never been published. At the prime 2, the computation of the differentials in the MSS for the Steenrod algebra through dimension 70 is described by Tangora [4]. In [5] May also introduced a general method (the May spectral sequence) for computing Ext groups over a Hopf algebra, by which Ravenel [6] computed $\mathrm{Ext}_{K(n)_*K(n)}^{s,*}(K(n)_*, K(n)_*)$ for $n \leqslant 2$ and $n = 3$, $p \geqslant 5$.

To our knowledge there are three versions of May spectral sequence for computing the Ext groups $\mathrm{Ext}_{A_*}^{s,*}(Z/p, Z/p)$ (cf. [5], [7] and [2, Theorem 3.2.5]). May's approach [5] is to filter the Steenrod algebra A rather than its dual and to study the resulting spectral sequence. He proved that the associated bi-graded algebra E_0A is primitively generated and its dual is isomorphic to

$$E^0A_* = E[\tau_i | i \geqslant 0] \otimes T[\xi_{i,j} | i > 0, j \geqslant 0], \tag{1.2}$$

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where $T[\ ]$ denotes the truncated polynomial algebra of height p on the indicated generators, and $E[\ ]$ denotes the exterior algebra. May [5] constructed an efficient complex

$$E_1 = E[h_{i,j}|i > 0, j \geqslant 0] \otimes P[b_{i,j}|i > 0, j \geqslant 0] \otimes P[a_i|i \geqslant 0] \tag{1.3}$$

(which is much smaller than the cobar complex) for computing the Ext groups over E^0A_*.

Zhou [7] actually gave an order in the generators of the cobar complex $C^{s,*}(A_*)$ rather than to filter it, from which he found an acyclic sub-complex $B^{s,*}(A_*)$ of $C^{s,*}(A_*)$ and proved that $C^{s,*}(A_*)/B^{s,*}(A_*)$ is isomorphic to E_1 in (1.3) as a Z/p-module. Thus the cohomology $H^{s,*}(C^{s,*}(A_*), d) = \mathrm{Ext}^{s,*}_{A_*}(Z/p, Z/p)$ is isomorphic to the cohomology of $C^{s,*}(A_*)/B^{s,*}(A_*)$.

In this paper, we follow Ravenel's ideal (cf. [2, Theorem 3.2.1]) but assign two inner degrees *the first inner degree* and *the second inner degree*, in the dual Steenrod algebra A_*. We filter the dual Steenrod algebra A_* and the cobar complex $C^s(A_*)$ by setting May filtration as $M(\tau_{i-1}) = M(\xi_i) = 2i - 1$ so that the resulting E^0A_* has the algebraic structure of (1.2), but the structure map $\Delta : E^0A_* \to E^0A_* \otimes E^0A_*$ is given by $\Delta(\xi_{i,j}) = \xi_{i,j} \otimes 1 + 1 \otimes \xi_{i,j}$ which is different from May's approach. We also set the E_0-term of the resulting spectral sequence (MSS) as $C^s(E^0A_*)$. Then the E_1-term is the cohomology $H^*(C^s(E^0A_*), d_0)$, which also has the algebraic structure of (1.3), from which, we discuss the properties of the MSS.

In Section 2, we introduce the concept of *sum of index* $SI(g)$ and *sum of digit* $Sd(g)$, which is very easy to compute. Then we prove a collapse theorem on the MSS:

Theorem 2.10 *If we have a cocycle $g = \sum \lambda_k g_k$ in May E_1-term $E_1^{s,t,b,M}$ such that $SI(g) = Sd(g)$, then g is a cocycle in May E_∞-term $E_\infty^{s,t,b,M}$.*

A easy consequence of the theorem is: *A cocycle in $E[h_{i,j}|i > 0, j \geqslant 0] \otimes P[a_i|i \geqslant 0]$ with homological dimension $< p$ is a cocycle in E_∞.*

Next in Section 3, we compute the higher May differentials of $b_{i,j}$ by virtue of the Adams-Novikov spectral sequence. In this section we also give some formulae for permuting the tensor product. Because the cobar complex is not commutative, the permutation of tensor product gives rise to higher May differentials. In Section 4, we define the *degree matrix* and the *degree equation* associated with generators of May E_1-term. Then detecting the May E_1-term is deduced to giving all the matrix solutions of the degree equation. The next section is devoted to the method to detect generators of May E_1-term $E_1^{s,t,b,*}$ for a given $(s, t, b, *)$. As the application we prove that some compositions of the known permanent cycles survive to E_∞ in the ASS in Section 6.

2 A Collapse Theorem in MSS

Hereafter we assume that p is an odd prime. Let A be the *mod p* Steenrod algebra. Then its dual A_* is isomorphic to

$$A_* = P[\xi_1, \xi_2, \ldots] \otimes E[\tau_0, \tau_1, \tau_2, \ldots].$$

In this paper we assign two inner degrees on A_*: *the first inner degree* of a monomial g in A_* denoted by $t(g)$, and *the second inner degree* of g denoted by $b(g)$. They are defined on each generator by

$$t(\xi_n) = t(\tau_n) = 2(p^n - 1),\ b(\xi_n) = 0 \text{ and } b(\tau_n) = 1. \tag{2.1}$$

Actually the element ξ_n could have the bi-degree $(2(p^n-1),0)$ as it has the degree $2(p^n-1)$ in graded algebra A_*; and one can put τ_n to have bi-degree $(2(p^n-1),1)$ because (1) τ_n has degree $2p^n-1$ in A_* and (2) there is a Bockstein occuring in the dual of τ_n as an element in the Steenrod algebra A. Thus the dual Steenrod algebra A_* becomes a bi-graded Hopf algebra, and the original inner degree is $t(g)+b(g)$.

Consider the cobar construction $C^s(A_*)=\bar{A}_*\otimes\cdots\otimes\bar{A}_*$ (with s tensor factors of the *augmentation ideal* $\bar{A}_*$). It is a trigraded cochain complex with differential $d: C^{s,t,b}(A_*)\to C^{s+1,t,b}(A_*)$. The differential d (cf. [5, 3.6]) is given by:

$$d(\alpha_1\otimes\cdots\otimes\alpha_s)=-\sum_{1\leqslant i\leqslant s}(-1)^{\lambda(i)}\alpha_1\otimes\cdots\otimes(\Delta(\alpha)-\alpha_i\otimes 1-1\otimes\alpha_i)\otimes\cdots\otimes\alpha_s, \tag{2.2}$$

where $\lambda(i)$ is the total degree of $\alpha_1\otimes\cdots\otimes\alpha_i'$ if $\Delta(\alpha_i)-\alpha_i\otimes 1-1\otimes\alpha_i=\sum\alpha_i'\otimes\alpha_i''$. Thus the cohomology $H^{s,t,b}(C(A_*),d)=\mathrm{Ext}^{s,t,b}_{A_*}(Z/p,Z/p)$ is trigraded, and the E_2-term of the classical Adams spectral sequence (ASS) becomes:

$$E_2^{s,t_0}=\oplus_{t+b=t_0}\mathrm{Ext}^{s,t,b}_{A_*}(Z/p,\ Z/p). \tag{2.3}$$

As Ravenel did in Theorem 3.2.5 of [2], we set May filtration on A_* by $M(\tau_{i-1})=M(\xi_i^{p^j})=2i-1$. The associated Hopf algebra $E^0A_*=F^MA_*/F^{M-1}A_*$ is trigraded with the algebra structure of (1.2), where τ_i is the projection of $\tau_i\in A_*$ and $\xi_{i,j}$ is the projection of $\xi_i^{p^j}$. Applying the May filtration to the cobar construction $C^s(A_*)$, we get a *four-filtrated May spectral sequence* (MSS) $(E_r^{s,t,b,M},d_r)\Longrightarrow\mathrm{Ext}^{s,t,b}_{A_*}(Z/p,Z/p)$. To describe the resulting spectral sequence, we have:

Theorem 2.4 [2, Theorem 3.2.5] *For $p>2$, the associated trigraded Hopf algebra E^0A_* is primitively generated with the algebra structure of* (1.2). *In the associated spectral sequence, the E_0-term $E_0^{s,t,b,M}$ is isomorphic to $C^{s,t,b,M}(E^0A_*)$ and the E_1-term $E_1^{s,t,b,M}=H^*(E_0,\ d_0)$ is isomorphic to*

$$E[h_{i,j}|i>0,j\geqslant 0]\otimes P[b_{i,j}|i>0,j\geqslant 0]\otimes P[a_i|i\geqslant 0],$$

where the homological dimension of each element is given by $s(a_i)=s(h_{i,j})=1,\ s(b_{i,j})=2$ and the degree is given by

$$h_{i,j}\in E_1^{1,2(p^i-1)p^j,0,2i-1},\quad b_{i,j}\in E_1^{2,2(p^i-1)p^{j+1},0,p(2i-1)}\quad\text{and}\quad a_i\in E_1^{1,2(p^i-1),1,2i+1},$$

here $h_{i,j}$ and a_i correspond respectively to $\xi_{i,j}$ and τ_i, $b_{i,j}$ corresponds to the summation $\sum_{0<k<p}\binom{p}{k}/p\ (\xi_{i,j}^k\otimes\xi_{i,j}^{p-k})$. One has $d_r: E_r^{s,t,b,M}\longrightarrow E_r^{s+1,t,b,M-r}$ and if $x\in E_r^{s,t,b,M}$ then

$$d_r(x\cdot y)=d_r(x)\cdot y+(-1)^s x\cdot d_r(y).$$

The first May differential d_1 is given by

$$d_1(h_{i,j})=\sum_{0<k<i}h_{i-k,k+j}h_{k,j},$$

$$d_1(a_i)=\sum_{0\leqslant k<i}h_{i-k,k}a_k$$

and

$$d_1(b_{i,j})=0.$$

For the product structure of the May spectral sequence, we have

Proposition 2.5 *In May E_1-term, we have the following relations*:

$$a_m h_{n,j} = h_{n,j} a_m, \quad h_{m,k} h_{n,j} = -h_{n,j} h_{m,k}, \quad a_m b_{n,j} = b_{n,j} a_m,$$
$$h_{m,k} b_{n,j} = b_{n,j} h_{m,k}, \quad a_m a_n = a_n a_m, \quad b_{m,n} b_{i,j} = b_{i,j} b_{m,n}.$$

We shall give the proof of this proposition in Section 3.

From now on, we denote $2(p-1)$ by q. We also use the symbols x, y and z to express the elements a_i, $h_{i,j}$ and $b_{i,j}$ respectively, thus the monomial of $E_1^{s,t,b,*}$ is denoted by

$$g = (x_1 \cdots x_b) \cdot (y_1 \cdots y_m) \cdot (z_1 \cdots z_l) \in E_1^{b+m+2l,t,b,*}. \tag{2.6}$$

For a monomial g in May E_1-term, we denote its homological dimension by $s(g)$, its first inner degree by $t(g)$, the second inner degree by $b(g)$ and its May filtration by $M(g)$ respectively.

Definition 2.7 *Define the sum of index on each element by*

$$SI(a_i) = SI(h_{i,j}) = i \quad \text{and} \quad SI(b_{i,j}) = 2i.$$

For a monomial g of the form (2.6), *define its sum of indices by*

$$SI(g) = \sum_{1 \leqslant i \leqslant b} SI(x_i) + \sum_{1 \leqslant i \leqslant m} SI(y_i) + \sum_{1 \leqslant i \leqslant l} SI(z_i).$$

For a linear sum of monomials $g = \sum \lambda_k g_k$, define $SI(g)$ by $SI(g) = \max\{SI(g_k)\}$.

Remark This definition is natural. For example, the monomial $a_m h_{i,j} b_{n,l}$ is represented in the cobar complex by $\tau_m \otimes \xi_i^{p^j} \otimes (\sum \binom{p}{k}/p \cdot \xi_n^{kp^l} \otimes \xi_n^{(p-k)p^l})$. We define its sum of indices to be $m + i + 2n$.

Definition 2.8 *If a positive integer t is divisible by q and the p-adic expression of t/q is given by $t/q = \bar{c}_0 + \bar{c}_1 p + \cdots + \bar{c}_n p^n$ with $0 \leqslant \bar{c}_i < p$, then we denote the sum of digits $\sum_{i \geqslant 0} \bar{c}_i$ by $Sd(t)$. For an element g of May E_1-term, we set $Sd(g) = Sd(t(g))$ for simplicity, where $t(g)$ is the first inner degree of g.*

For example, we see that

$$t(a_i)/q = 1 + p + \cdots + p^{i-1},$$
$$t(h_{i,j})/q = p^j + p^{j+1} + \cdots + p^{i+j-1},$$
$$t(b_{i,j})/q = p^{j+1} + p^{j+2} + \cdots + p^{i+j}$$

and then $Sd(a_i) = i$ and $Sd(h_{i,j}) = Sd(b_{i,j}) = i$.

Lemma 2.9 *If g is a monomial of the form* (2.6), *then we have*:

(1) *The May filtration $M(g) \geqslant 2(SI(g) + b(g)) - s(g)$ and the equality holds if and only if*

$$g \in P[a_i | i \geqslant 0] \otimes E[h_{i,j} | i > 0, j \geqslant 0].$$

(2) *$SI(g) \geqslant Sd(g)$, and the equality holds if $g \in P[a_i | i \geqslant 0] \otimes E[h_{i,j} | i > 0, j \geqslant 0]$ and $s(g) < p$.*

Proof (1) Note the facts that

$$M(a_i) = 2i + 1 = 2(SI(a_i) + b(a_i)) - s(a_i),$$
$$M(h_{i,j}) = 2i - 1 = 2(SI(h_{i,j} + b(h_{i,j})) - s(h_{i,j}),$$
$$M(b_{i,j}) = p(2i - 1) > 2(SI(b_{i,j}) + b(b_{i,j})) - s(b_{i,j}).$$

It is easy to get (1).

(2) From the facts that $SI(b_{i,j}) = 2i = 2Sd(b_{i,j})$, $SI(h_{i,j}) = i = Sd(h_{i,j})$ and $SI(a_i) = i = Sd(a_i)$, for any g which is a monomial of the form (2.6) we can easily have

$$SI(g) \geqslant Sd(g).$$

Suppose that $g \in P[a_i|i \geqslant 0] \otimes E[h_{i,j}|i > 0, j \geqslant 0]$ and $s(g) < p$. We can assume that $g = (a_{j_1}a_{j_2}\cdots a_{j_r}) \cdot (h_{i_1,l_1}h_{i_2,l_2} \cdot h_{i_k,l_k})$, where $s(g) = r + k < p$. We can get

$$SI(g) = \sum_{1\leqslant i\leqslant r} SI(a_{j_i}) + \sum_{1\leqslant j\leqslant k} SI(h_{i_j,l_j}) = \sum_{1\leqslant i\leqslant r} j_i + \sum_{1\leqslant j\leqslant k} i_j.$$

Now we consider $Sd(g)$. By definition, we have

$$Sd(g) = Sd(t(g))$$

and

$$t(g) = \sum_{1\leqslant i\leqslant r} t(a_{j_i}) + \sum_{1\leqslant j\leqslant k} t(h_{i_j,l_j}).$$

It follows that

$$t(g)/q = \sum_{1\leqslant i\leqslant r} t(a_{j_i})/q + \sum_{1\leqslant j\leqslant k} t(h_{i_j,l_j})/q.$$

From the examples above Lemma 2.9, we can have

$$t(g)/q = \sum_{1\leqslant i\leqslant r} (1 + p + \cdots + p^{j_i-1}) + \sum_{1\leqslant j\leqslant k} (p^{l_j} + p^{l_j+1} + \cdots + p^{i_j+l_j-1}).$$

Note that $r + k < p$. By the definition of the sum digit and the knowledge on p-adic expression in number theory, we can also obtain that

$$Sd(g) = Sd(t(g)) = \sum_{1\leqslant i\leqslant r} j_i + \sum_{1\leqslant j\leqslant k} i_j.$$

Thus we have $SI(g) = Sd(g)$. The proof of (2) is completed.

From the discussion above, we get a collapse theorem on the MSS.

Theorem 2.10 *If we have a cocycle $g = \sum \lambda_k g_k$ in May E_1-term $E_1^{s,t,b,M}$ such that $SI(g) = Sd(g)$, then g is a cocycle in May E_∞-term $E_\infty^{s,t,b,M}$.*

Proof Consider the May differential $d_r : E_r^{s,t,b,M} \to E_r^{s+1,t,b,M-r}$, we see that $d_1(g) = 0$. Then we get the theorem from $E_r^{s+1,t,b,M-r} = 0$ for $r \geqslant 2$. Indeed from $SI(g) = Sd(g)$ and $SI(b_{i,j}) = 2Sd(b_{i,j})$, we see that $g \in P[a_i|i \geqslant 0] \otimes E[h_{i,j}|i > 0, j \geqslant 0]$ and then $M = 2(Sd(t) + b) - s$. Suppose we have a monomial g' of the form (2.6) in $E_1^{s+1,t,b,M-r}$. Then we have $s(g') = s + 1$, $Sd(g') = Sd(t)$, $b(g') = b$ and $M(g') = M - r$. By Lemma 2.9, we have $M(g') = M - r \geqslant 2(Sd(t) + b) - (s + 1)$, which contradicts $r \geqslant 2$.

Corollary 2.11 *A cocycle g in $P[a_i|i \geqslant 0] \otimes E[h_{i,j}|i > 0, j \geqslant 0] \subset E_1$ with homological dimension $s(g) < p$ is permanent.*

Proof The cocycle g has the property $SI(g) = Sd(g)$ by Lemma 2.9, (2).

For example,

$$\begin{aligned} &a_1^s h_{2,0} h_{1,0} && \text{for } s < p-2, \\ &a_2^s h_{2,0} h_{1,1} && \text{for } s < p-2, \\ &h_{n,j} h_{n-1,j} \cdots h_{1,j} && \text{and} \\ &h_{n,j} h_{n-1,j+1} \cdots h_{1,j+n-1} && \text{for } n < p \end{aligned}$$

are cocycles in $P[a_i|i \geqslant 0] \otimes E[h_{i,j}|i > 0, j \geqslant 0]$. Thus they are cocycles in May E_∞.

We also point out that all the following generators

$$\begin{array}{llll} h_{2,i} h_{1,i} & h_{2,i} h_{1,i+1} & h_{3,i} h_{2,i} h_{1,i} & h_{2,i} h_{2,i-1} h_{1,i} \\ h_{3,0} h_{1,2} h_{1,0} & h_{3,0} h_{2,1} h_{1,2} & h_{3,0} h_{2,1} h_{2,0} h_{1,1} & h_{4,0} h_{3,0} h_{2,0} h_{1,0} \\ h_{3,1} h_{2,1} h_{2,0} h_{1,1} & h_{3,0} h_{2,2} h_{1,2} h_{1,0} & & \end{array}$$

of $H^{*,*}(U(L))$ given in (2.5) of [8] are permanent because they all have the property $SI(g) = Sd(g)$.

3 Some Properties of the Higher May Differential

In this section we shall study the properties of higher May differentials. To do this we are required to work back in the cobar complex $C^{s,*}(A_*)$. The point we start is the higher May differential of $b_{i,j}$.

Lemma 3.1 *For $i > 1$, we have the cochain $\widetilde{b}_{i,j}$ in the cobar complex $C^{2,*}(A_*)$ which is sent to $b_{i,j}$ in May E_1-term and*

$$d(\widetilde{b}_{i,j}) = -\sum_{0<k<i} \widetilde{b}_{i-k,j+k} \otimes \xi_k^{p^{j+1}} + \sum_{0<k<i} \xi_{i-k}^{p^{j+k+1}} \otimes \widetilde{b}_{k,j}.$$

Thus in the MSS we have

$$d_{2p-1}(b_{i,j}) = -\, b_{i-1,j+1} h_{1,j+1} + h_{1,i+j} b_{i-1,j}.$$

Proof In $C^{2,*}(A_*)$, define

$$\begin{aligned} p \cdot \widetilde{b}_{i,j} = &\Big(\xi_i \otimes 1 + \sum_{0<k<i} \xi_{i-k}^{p^k} \otimes \xi_k + 1 \otimes \xi_i\Big)^{p^{j+1}} \\ &- \xi_i^{p^{j+1}} \otimes 1 - \sum_{0<k<i} \xi_{i-k}^{p^{j+k+1}} \otimes \xi_k^{p^{j+1}} - 1 \otimes \xi_i^{p^{j+1}}. \end{aligned}$$

It is easy to see that $\widetilde{b}_{i,j}$ is sent to $b_{i,j}$ in May E_1-term.

Consider the Thom map $\Phi : BP \to KZ/p$ and the induced Thom maps $\Phi : BP_* \to Z/p$ and $\Phi : BP_*BP \to A_*$. Applying the cobar construction, we get the Thom reductions

$$\Phi : C^{s,*}(BP_*BP) \longrightarrow C^{s,*,*}(A_*)$$

and

$$\Phi : \mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, BP_*) \longrightarrow \mathrm{Ext}^{s,*,*}_{A_*}(Z/p, Z/p).$$

Notice that the differential $d : C^{s,*}(BP_*BP) \to C^{s+1,*}(BP_*BP)$ is defined by $d(x) = x \otimes 1 + 1 \otimes x - \Delta(x)$ for $x \in C^{1,*}(BP_*BP)$ (cf. [9, (1.10)]), we have $\Phi \cdot d = -d \cdot \Phi$.

Recall from [2, (4.1.18)] that $BP_* = Z_{(p)}[v_1, v_2, \ldots, v_n, \ldots]$ and $BP_*BP = BP_*[t_1, t_2, \ldots]$ with $|v_n| = |t_n| = 2(p^n - 1)$. The structure map (cf. [2, (4.3.15)]) $\Delta : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$ is given by

$$\Delta(t_i) = \sum_{0 \leqslant k \leqslant i} t_k \otimes t_{i-k}^{p^k} \bmod (p, v_1, v_2, \ldots) = I. \tag{3.2}$$

The Thom map Φ sends v_i to 0 and t_i to $c(\xi_i)$ in A_*, where $c : A_* \to A_*$ is the conjugation. Thus $\Phi(c(t_i)) = \xi_i$, where c is the conjugation of BP_*BP.

To compute $d(\widetilde{b}_{i,j})$, consider the differential $d(c(t_i^{p^{j+1}}))$. Let

$$p \cdot \bar{b}_{i,j} = \Bigg(\sum_{0 \leqslant k \leqslant i} c(t_{i-k}^{p^k}) \otimes c(t_k) \Bigg)^{p^{j+1}} - \sum_{0 \leqslant k \leqslant i} c(t_{i-k}^{p^{k+j+1}}) \otimes c(t_k^{p^{j+1}}).$$

Then from the commutativity of the following diagram

$$\begin{array}{ccc} BP_*BP & \xrightarrow{\Delta} & BP_*BP \otimes_{BP_*} BP_*BP \\ \Big\downarrow c & & \Big\downarrow (c\otimes c)\cdot T \\ BP_*BP & \xrightarrow{\Delta} & BP_*BP \otimes_{BP_*} BP_*BP \end{array}$$

and (3.2), we see that mod $(p^2, pv_1, pv_2, \ldots, v_1^p, v_2^p, \ldots)$

$$d(c(t_i^{p^{j+1}})) \equiv -p \cdot \bar{b}_{i,j} - \sum_{0<k<i} c(t_{i-k}^{p^{k+j+1}}) \otimes c(t_k^{p^{j+1}}),$$

$$d(d(c(t_i^{p^{j+1}}))) \equiv -p \cdot d\left(\bar{b}_{i,j}\right) + \sum_{0<k<i} p \cdot \bar{b}_{i-k,j+k} \otimes c(t_k^{p^{j+1}}) - \sum_{0<k<i} p \cdot c(t_{i-k}^{p^{k+j+1}}) \otimes \bar{b}_{k,j} \equiv 0.$$

Thus in $C^{2,*}(BP_*BP)$, we have

$$d\left(\bar{b}_{i,j}\right) \equiv \sum_{0<k<i} \bar{b}_{i-k,j+k} \otimes c(t_k^{p^{j+1}}) - \sum_{0<k<i} c(t_{i-k}^{p^{k+j+1}}) \otimes \bar{b}_{k,j}.$$

Applying the Thom reduction, we get the lemma from $\Phi(\bar{b}_{i,j}) = \widetilde{b}_{i,j}$.

In the cobar complex, the tensor product is not commutative, thus permuting the tensor product will give rise to higher May differentials. Here we give some formulae on permuting the tensor order in the cobar complex of $P[\xi_1, \xi_2, \ldots] \subset A_*$. There are the similar formulae in the cobar complex of A_*, but because of the formula (2.2), the sign becomes complicated if τ_i appears.

Lemma 3.3 *For $x, y \in P[\xi_1, \xi_2, \ldots]$, we have*

$$d(x \cdot y) = x \otimes y + y \otimes x + d(x) \cdot \Delta(y) + (x \otimes 1 + 1 \otimes x) \cdot d(y).$$

Thus in the MSS we have

$$x \otimes y = -\, y \otimes x - d(x) \cdot \Delta(y) - (x \otimes 1 + 1 \otimes x) \cdot d(y)$$

and the higher May differentials come from $-d(x)\cdot\Delta(y)-(x\otimes 1+1\otimes x)\cdot d(y)$.

Proof Notice that for α in $P[\xi_1,\xi_2,\ldots]$, the total degree of α' in $\Delta(\alpha)=\sum\alpha'\otimes\alpha''$ is 1 mod (2) all the time. Thus we have

$$\Delta(\alpha)=\alpha\otimes 1+1\otimes\alpha+d(\alpha),$$

and then

$$\begin{aligned}d(x\cdot y)=&\Delta(x)\cdot\Delta(y)-x\cdot y\otimes 1-1\otimes x\cdot y\\ =&(x\otimes 1+1\otimes x+d(x))\cdot(y\otimes 1+1\otimes y+d(y))-x\cdot y\otimes 1-1\otimes x\cdot y\\ =&x\otimes y+y\otimes x+d(x)\cdot\Delta(y)+(x\otimes 1+1\otimes x)\cdot d(y).\end{aligned}$$

Lemma 3.4 *For* x,y *and* u *in* $P[\xi_1,\xi_2,\ldots]$, *we have*

$$\begin{aligned}d\left((x\otimes y)\cdot\Delta(u)\right)=&-x\otimes y\otimes u+u\otimes x\otimes y+d(x\otimes y)\cdot(\Delta^2(u))\\ &+(1\otimes x\otimes y)\cdot(1\otimes\Delta)d(u)-(x\otimes y\otimes 1)\cdot(\Delta\otimes 1)d(u).\end{aligned}$$

Proof Denote $\Delta(u)=u\otimes 1+1\otimes u+\sum u'\otimes u''$. Then $d(u)=\sum u'\otimes u''$ and

$$(x\otimes y)\cdot\Delta(u)=x\cdot u\otimes y+x\otimes y\cdot u+\sum x\cdot u'\otimes y\cdot u''.$$

Notice that

$$\begin{aligned}\Delta^2(u)=&(\Delta\otimes 1)\Delta(u)\\ =&\sum\Delta(u')\otimes u''+\sum u'\otimes u''\otimes 1+1\otimes 1\otimes u+1\otimes u\otimes 1+u\otimes 1\otimes 1\end{aligned}$$

and

$$\begin{aligned}\Delta^2(u)=&(1\otimes\Delta)\Delta(u)\\ =&\sum u'\otimes\Delta(u'')+\sum 1\otimes u'\otimes u''+1\otimes 1\otimes u+1\otimes u\otimes 1+u\otimes 1\otimes 1.\end{aligned}$$

We see that

$$\sum\Delta(u')\otimes u''+\sum u'\otimes u''\otimes 1=\sum u'\otimes\Delta(u'')+\sum 1\otimes u'\otimes u''. \tag{3.5}$$

Then

$$\begin{aligned}(\Delta\otimes 1)d(u)&=\sum\Delta(u')\otimes u''\\ &=\sum u'\otimes\Delta(u'')-\sum u'\otimes u''\otimes 1+\sum 1\otimes u'\otimes u'' \qquad \text{by (3.5)}\\ &=\sum u'\otimes d(u'')+\sum u'\otimes 1\otimes u''+1\otimes d(u) \qquad (3.6)\end{aligned}$$

and

$$\begin{aligned}(1\otimes\Delta)d(u)&=\sum u'\otimes\Delta(u'')\\ &=\sum\Delta(u')\otimes u''-\sum 1\otimes u'\otimes u''+\sum u'\otimes u''\otimes 1 \qquad \text{by (3.5)}\\ &=\sum d(u')\otimes u''+\sum u'\otimes 1\otimes u''+d(u)\otimes 1. \qquad (3.7)\end{aligned}$$

Applying Lemma 3.3, we get the lemma from the following computations:

$$\begin{aligned} d(x\cdot u\otimes y) &= d(x\cdot u)\otimes y - x\cdot u\otimes d(y) = u\otimes x\otimes y + x\otimes u\otimes y \\ &\quad - \underline{(x\otimes d(y))\cdot(u\otimes 1\otimes 1)}_2 + \underline{(d(x)\otimes y)\cdot(\Delta(u)\otimes 1)}_1 \\ &\quad + \underline{(x\otimes 1\otimes y)\cdot(d(u)\otimes 1)}_{01} + \underline{(1\otimes x\otimes y)\cdot(d(u)\otimes 1)}_4, \\ d(x\otimes y\cdot u) &= d(x)\otimes y\cdot u - x\otimes d(y\cdot u) \\ &= \underline{(d(x)\otimes y)\cdot(1\otimes 1\otimes u)}_1 - x\otimes u\otimes y - x\otimes y\otimes u \\ &\quad - \underline{(x\otimes d(y))\cdot(1\otimes\Delta(u))}_2 - \underline{(x\otimes y\otimes 1)\cdot(1\otimes d(u))}_3 \\ &\quad - \underline{(x\otimes 1\otimes y)\cdot(1\otimes d(u))}_{02}, \\ \sum d(x\cdot u'\otimes y\cdot u'') &= \sum d(x\cdot u')\otimes y\cdot u'' - \sum x\cdot u'\otimes d(y\cdot u'') \\ &= \sum \underline{(x\otimes 1\otimes y)(1\otimes u'\otimes u'')}_{02} + \sum \underline{(1\otimes x\otimes y)(u'\otimes 1\otimes u'')}_4 \\ &\quad + \sum \underline{(d(x)\otimes y)(\Delta(u')\otimes u'')}_1 + \sum \underline{(x\otimes 1\otimes y)(d(u')\otimes u'')}_{03} \\ &\quad + \sum \underline{(1\otimes x\otimes y)(d(u')\otimes u'')}_4 \\ &\quad - \sum \underline{(x\otimes y\otimes 1)(u'\otimes 1\otimes u'')}_3 - \sum \underline{(x\otimes 1\otimes y)(u'\otimes u''\otimes 1)}_{01} \\ &\quad - \sum \underline{(x\otimes d(y))(u'\otimes\Delta(u''))}_2 - \sum \underline{(x\otimes y\otimes 1)(u'\otimes d(u''))}_3 \\ &\quad - \sum \underline{(x\otimes 1\otimes y)(u'\otimes d(u''))}_{03}, \end{aligned}$$

where the terms with subscript 01, 02 and 03 amount to zero by $d(d(u)) = \sum d(u')\otimes u'' - \sum u'\otimes d(u'') = 0$, the terms with subscript 1 and 2 amount to

$$(d(x)\otimes y)\cdot(\Delta\otimes 1)\Delta(u) - (x\otimes d(y))\cdot(1\otimes\Delta)\Delta(u) = d(x\otimes y)\cdot\Delta^2(u)$$

and the terms with subscript 3 and 4 respectively amount to $-(x\otimes y\otimes 1)\cdot(\Delta\otimes 1)d(u)$ and $(1\otimes x\otimes y)\cdot(1\otimes\Delta)d(u)$ by (3.6) and (3.7).

Lemma 3.8 *For x, y and u, v in $P[\xi_1, \xi_2, \ldots]$, we have*

$$\begin{aligned} &d\left((x\otimes y)\cdot\Delta(u)\otimes v - u\otimes(x\otimes y)\cdot\Delta(v)\right) \\ &\quad = -x\otimes y\otimes u\otimes v + u\otimes v\otimes x\otimes y \\ &\qquad + (d(x\otimes y)\otimes 1)\cdot(\Delta^2(u)\otimes v) + (1\otimes d(x\otimes y))\cdot(u\otimes\Delta^2(v)) \\ &\qquad + (1\otimes x\otimes y\otimes 1)\cdot(1\otimes\Delta\otimes 1)d(u\otimes v) \\ &\qquad - (1\otimes 1\otimes x\otimes y)\cdot(1\otimes 1\otimes\Delta)d(u\otimes v) - (x\otimes y\otimes 1\otimes 1)\cdot(\Delta\otimes 1\otimes 1)d(u\otimes v). \end{aligned}$$

Proof Applying Lemma 3.4, the lemma follows from direct computations.

Proof of Proposition 2.5 The formulae $h_{m,k}h_{n,j} = -h_{n,j}h_{m,k}$, $h_{m,k}b_{n,j} = b_{n,j}h_{m,k}$ and $b_{m,n}b_{i,j} = b_{m,n}b_{i,j}$ follow from Lemmas 3.3, 3.4 and 3.8 respectively. Similarly we get the others by computing $d(\xi_n^{p^j}\cdot\tau_m)$, $d(\widetilde{b}_{n,j}\cdot\Delta(\tau_m))$ and $d(\tau_m\cdot\tau_n)$.

4 The Matrix Associated with the First Inner Degree

In this section we define the *degree matrix* and the *degree equation* associated with the generator g of May E_1-term.

Suppose we have a monomial $g = (x_1 \cdots x_b) \cdot (y_1 \cdots y_m) \cdot (z_1 \cdots z_l)$ of the form (2.6). We say that g has (b, m, l)-type. Notice that the first inner degree of x_i, y_i and z_i could be uniquely expressed as

$$\begin{aligned} t(x_i) &= q(x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n), \\ t(y_i) &= q(y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n), \\ t(z_i) &= q(0 + z_{i,1}p + \cdots + z_{i,n}p^n) \end{aligned}$$

and the digit sequence $(x_{1,0}, x_{1,1}, \ldots, x_{1,n})$ is the form of $(1 \cdots 1, 0 \cdots 0)$, while $(y_{i,0}, \ldots, y_{i,n})$ and $(0, z_{i,1}, \ldots, z_{i,n})$ are of the form $(0 \cdots 0, 1 \cdots 1, 0 \cdots 0)$. Denote the sequences by columns. Then the generator g determines a matrix

$$\begin{pmatrix} x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} & 0 & \cdots & 0 \\ x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} & z_{1,1} & \cdots & z_{l,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n} & z_{1,n} & \cdots & z_{l,n} \end{pmatrix} \begin{matrix} c_0 \\ c_1 \\ \vdots \\ c_n. \end{matrix} \tag{4.1}$$

We call (4.1) the *degree matrix* of (b, m, l)-*type* associated with g. By the commutativity of $E_1^{s,t,b,*}$, the monomial g is arranged in the following way:

(a) If $i > j$, we put a_i on the left side of a_j.

(b) We put $h_{i,j}$ on the left side of $h_{m,k}$ if $j < k$.

(c) If $i > m$, we put $h_{i,j}$ on the left side of $h_{m,j}$.

(d) Apply the same rules (b) and (c) to $b_{i,j}$.

Thus the entries of the degree matrix (4.1) are 0 or 1, and satisfy:

$$\begin{cases} (1)\ x_{1,j} \geqslant x_{2,j} \geqslant \cdots \geqslant x_{b,j},\ x_{i,0} \geqslant x_{i,1} \geqslant \cdots \geqslant x_{i,n} \text{ for } i \leqslant b,\ j \leqslant n. \\ (2)\ \text{If } y_{i,j-1} = 0 \text{ and } y_{i,j} = 1 \text{ then for all } k < j,\ y_{i,k} = 0. \\ (3)\ \text{If } y_{i,j} = 1 \text{ and } y_{i,j+1} = 0 \text{ then for all } k > j,\ y_{i,k} = 0. \\ (4)\ y_{1,0} \geqslant y_{2,0} \geqslant \cdots \geqslant y_{m,0}. \\ (5)\ \text{If } y_{i,0} = y_{i+1,0},\ y_{i,1} = y_{i+1,1},\ \ldots,\ y_{i,j} = y_{i+1,j}, \text{ then } y_{i,j+1} \geqslant y_{i+1,j+1}. \\ (6)\ \text{Apply the same rules } (2) \sim (5) \text{ to } z_{i,j}. \end{cases} \tag{4.2}$$

Suppose that the degree of the monomial g is $\{s, t, b, *\}$ and t/q is expressed by p-adic number as $t/q = \bar{c}_0 + \bar{c}_1 p + \cdots + \bar{c}_n p^n$. Consider the *sum of rows* in the degree matrix (4.1), we have the following equation

$$\begin{cases} \sum\limits_{1 \leqslant i \leqslant b} x_{i,0} + \sum\limits_{1 \leqslant i \leqslant m} y_{i,0} = c_0, \\ \sum\limits_{1 \leqslant i \leqslant b} x_{i,1} + \sum\limits_{1 \leqslant i \leqslant m} y_{i,1} + \sum\limits_{1 \leqslant i \leqslant l} z_{i,1} = c_1, \\ \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ \sum\limits_{1 \leqslant i \leqslant b} x_{i,n-1} + \sum\limits_{1 \leqslant i \leqslant m} y_{i,n-1} + \sum\limits_{1 \leqslant i \leqslant l} z_{i,n-1} = c_{n-1}, \\ \sum\limits_{1 \leqslant i \leqslant b} x_{i,n} + \sum\limits_{1 \leqslant i \leqslant m} y_{i,n} + \sum\limits_{1 \leqslant i \leqslant l} z_{i,n} = c_n. \end{cases} \tag{4.3}$$

Then from the properties of the p-adic number we see that

$$\begin{aligned} c_0 &= \bar{c}_0 + k_1 p, \\ c_1 &= \bar{c}_1 - k_1 + k_2 p, \\ &\cdots \\ c_{n-1} &= \bar{c}_{n-1} - k_{n-1} + k_n p, \\ c_n &= \bar{c}_n - k_n. \end{aligned} \tag{4.4}$$

We call (4.3) the *degree equation of type* (b, m, l) associated with g. The number sequence $\mathbf{k} = (k_1, k_2, \dots, k_n)$ in (4.4) is called the *carry sequence*. The sequence $\mathbf{c} = (c_0, c_1, \dots, c_n)$ which is determined by $(\bar{c}_0, \bar{c}_1, \dots, \bar{c}_n)$ and the carry sequence is called the *sum of row sequences*.

Remark 4.5 On the degree matrix (4.1) and degree equation (4.3), we remark the following:

(1) n is the maximal digit number in the p-adic expression of t/q.

(2) If we denote the width of the matrix (4.1) by $b+m+l$, then $b+m+l$ is less than the homological dimension s unless $l=0$. Observe the degree equation, we also find that $\max\{c_i\} \leqslant b+m+l$.

Notice that the elements x_i, y_i and z_i are uniquely determined by their first inner degrees. The matrix solution of (4.3) satisfying (4.2) determines a monomial g of type (b, m, l) by setting the first inner degree $t(x_i)$ (resp. $t(y_i)$ and $t(z_i)$) to be $q(x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n)$ (resp. $q\sum_k y_{i,k}p^k$ and $q\sum_k z_{i,k}p^k$). Thus for a given $\{s, t, b, *\}$, the determination of May E_1-term $E_1^{s,t,b,*}$ is deduced to:

(1) List up all the possible types (b, m, l) so that $b+m+2l = s$.

(2) For each given type (b, m, l), list up all the carry sequences $\mathbf{k} = (k_1, \dots, k_n)$ such that $\max\{c_i\} \leqslant b+m+l$.

(3) For each given type (b, m, l) and carry sequence $\mathbf{k}$, solve the corresponding degree equation. Then determine all the generators of $E_1^{s,t,b,*}$ from the matrix solutions.

5 Detecting Generators in May E_1-term

For a given triple $\{s, t, b\}$, detecting the generators of $E_1^{s,t,b,*}$ is still complicated although it is deduced to solving the degree equations. In this section we will simplify the method further.

With the notation in Section 4, choose a carry sequence $\mathbf{k}$, the sum of row sequence $\mathbf{c}$ is determined by (4.4). For the sequence $\mathbf{c}$, we define

$$\begin{aligned} &m_0 = \max\{c_0 - b, 0\}, \quad m_i = \max\{c_i - c_{i-1}, 0\} \text{ for } i > 0 \text{ and} \\ &\widetilde{m} = m_0 + m_1 + \cdots + m_n. \end{aligned} \tag{5.1}$$

Consider the $(b, m, 0)$-type degree equation

$$\begin{cases} x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{m,0} = c_0, \\ x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{m,1} = c_1, \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{m,n} = c_n. \end{cases} \tag{5.2}$$

We have the following *Simplest way* to construct its matrix solutions.

Simplest way 5.3 *We may construct the matrix solutions of* (5.2) *as follows* :

(1) *Set the first row as* $(1,\dots,1,0,\dots,0)$ *with* c_0 $1's$.

Inductively suppose we have set the first i *rows as*

$$\left(\begin{array}{ccc|ccccc} * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \end{array}\right) \begin{array}{l} c_0 \\ \vdots \\ c_{i-1}. \end{array}$$

Then set the $(i+1)^{st}$ *row as* :

(2) *In the case* $c_i \geqslant c_{i-1}$, *start the* $1's$ *in the next neighboring* $c_i - c_{i-1}$ *columns of the* $y_{t,i}$ *part, like*

$$\left(\begin{array}{ccc|ccc|ccc|ccc} & & & & & & & b & & & & \\ * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline * & \cdots & * & * & \cdots & * & 1 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right) \begin{array}{l} \\ c_0 \\ \vdots \\ c_{i-1} \\ \hline c_i. \end{array}$$

(3) *In the case* $c_i < c_{i-1}$, *stop the* $1's$ *in some former* $c_{i-1} - c_i$ *columns as*

$$\left(\begin{array}{ccccccc|ccc} * & \cdots & * & \cdots & * & \cdots & * & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & 1 & \cdots & 1 & \cdots & * & 0 & \cdots & 0 \\ \hline * & \cdots & 0 & \cdots & 0 & \cdots & * & 0 & \cdots & 0 \end{array}\right) \begin{array}{l} c_0 \\ \vdots \\ c_{i-1} \\ \hline c_i. \end{array}$$

In this case we have different choices if $c_i \neq 0$.

For example, to solve the $(s-3,4,0)$-type degree equation with sum of row sequence $\mathbf{c} = (s, s+1, s)$ by the Simplest way 5.3, we see that the first two rows are

$$\left(\begin{array}{cccc|cccc} 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} s \\ s+1. \end{array}$$

Then the possible third rows are

$$\left(\begin{array}{cccc|cccc} 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & \cdots & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{lll} s & & \\ s+1 & & \\ \hline s & \cdots & (1) \\ & \cdots & (2) \\ & \cdots & (3). \end{array}$$

If we choose (1), (2) and (3) as the third row respectively, then we get the following three solutions:

$$\left(\begin{array}{ccc|cccc} 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 \end{array}\right) \begin{array}{l} s \\ s+1 \\ s, \end{array}$$

$$\left(\begin{array}{ccc|cccc} 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} s \\ s+1 \\ s, \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdots & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} s \\ s+1 \\ s. \end{array}$$

Unfortunately the matrix solutions got by the Simplest way 5.3 do not always detect generators of $E_1^{b+m,*,b,*}$. For example the three solutions above detect $a_3^{s-3}h_{3,0}^3h_{1,1}$, $a_3^{s-3}h_{3,0}^2h_{2,0}h_{2,1}$ and $a_3^{s-4}a_2h_{3,0}^3h_{2,1}$ respectively. For this reason we define $F_1^{s,t,b,*}$ to be the algebra

$$P[a_i|i \geqslant 0] \otimes P[b_{i,j}|i > 0, j \geqslant 0] \otimes P[h_{i,j}|i > 0, j \geqslant 0],$$

and we have the obvious identification $E_1^{s,t,b,*} = F_1^{s,t,b,*}/(h_{i,j}^2)$.

Remark 5.4 For the matrix solutions got by the Simplest way 5.3 we remark the following:

(1) In a matrix solution, if all the entries of a column are 0 in the $y_{i,j}$ parts, it deduces none because $t(h_{i,j}) > 0$. But it deduces a_0 while in the $x_{i,j}$ part.

(2) If $m < \widetilde{m}$ where $\widetilde{m}$ is defined in (5.1), then the $(b, m, 0)$-type degree equation has no solution.

(3) If $m > \widetilde{m}$, then any matrix solution has all entries in the last column of the $y_{i,j}$ parts being 0 and then it deduces none.

For an element $g = x_1 \cdots x_b \cdot y_1 \cdots y_m$ in $F_1^{b+m,t,b,*}$, we denote the set of terms in $d_1(g)$ by $D_1\{g\}$. For instance, we have

$$D_1\{a_i\} = \{a_0h_{i,0}, a_1h_{i-1,1}, \ldots, a_{i-1}h_{1,i-1}\}$$

and

$$D_1\{h_{i,j}\} = \{h_{1,j}h_{i-1,j+1}, h_{2,j}h_{i-2,j+2}, \ldots, h_{i-1,j}h_{1,i+j-1}\}.$$

Then $D_1\{g\}$ generates a submodule of $F_1^{b+m+1,t,b,*}$ and $D_1^k\{g\} = D_1\{\cdots D_1\{g\}\cdots\}$ generates a submodule of $F_1^{b+m+k,t,b,*}$.

Lemma 5.5 *For a given sum of row sequence* $\mathbf{c}$*, any monomial* g' *of* $(b, m+1, 0)$*-type is detected in* $D_1\{g\}$ *for some appropriate* $g \in F_1^{b+m,t,b,*}$ *if the corresponding* $(b, m, 0)$*-type degree equation has solutions. Thus any monomial of* $(b, \widetilde{m}+k, 0)$*-type is detected in* $D_1^k\{g\}$ *for some* $g \in F_1^{b+\widetilde{m},t,b,*}$ *and* g *is deduced by the Simplest way* 5.3.

Proof Let $g' = x_1 \cdots x_b \cdot y_1 \cdots y_m y_{m+1}$ be a monomial induced by a solution of $(b, m+1, 0)$-type degree equation with sum row sequence $\mathbf{c}$. Then $g' \in D_1\{g\}$ is equivalent to saying that in the degree matrix of g' there are two columns being divided from one, like the left one in the following

$$\begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \begin{matrix} j' \\ \vdots \\ j-1 \\ j \\ \vdots \\ j+k \end{matrix} \Rightarrow \begin{pmatrix} & \vdots & & \\ \cdots & 1 & \cdots & \cdots \\ & \vdots & & \\ \cdots & 1 & \cdots & \cdots \\ \cdots & 1 & \cdots & \cdots \\ & \vdots & & \\ \cdots & 1 & \cdots & \cdots \\ & \vdots & & \end{pmatrix} \begin{matrix} j' \\ \vdots \\ j-1 \\ j \\ \vdots \\ j+k. \end{matrix} \tag{5.6}$$

In this case, glue the two columns into one like the right matrix. Then the resulting matrix induces a generator g of $(b,m,0)$-type in $F_1^{b+m,b,t,*}$ and $g' \in D_1\{g\}$.

For the sum of row sequence $\mathbf{c}$, the corresponding $(b,m,0)$-type degree equation having solutions assures us $m \geqslant \widetilde{m}$ and then $m+1 > \widetilde{m}$. Thus the degree matrix of g' is not constructed by the Simplest way 5.3 (by Remark 5.4, 3) and one of the following cases must appear.

(1) The first row is of the form

$$\left(\begin{array}{cccc|cc} 1\cdots 1 & 0 & \cdots & \underset{b}{0} & 1\cdots 1 & 0\cdots 0 \end{array} \right) \; c_0.$$

In this case, $g = \cdots a_0 \cdot h_i \cdots$ and $g \in D_1\{x_1 \cdots a_i \cdots y_m\}$.

(2) In the case $c_i \geqslant c_{i-1}$, start the $1's$ in more than $c_i - c_{i-1}$ columns. In this case, we have to stop some $1's$ in the former columns like

$$\left(\begin{array}{ccccc|ccc|c} * & \cdots & * & \cdots & * & 0 & \cdots & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & \\ * & \cdots & 1 & \cdots & * & 0 & \cdots & 0 & \cdots \\ \hline * & \cdots & 0 & \cdots & * & 1 & \cdots & 1 & \cdots \end{array} \right) \begin{array}{c} c_0 \\ \vdots \\ c_{i-1} \\ \hline c_i. \end{array}$$

It is of the form (5.6).

(3) In the case $c_i < c_{i-1}$, stop more than $(c_{i-1} - c_i)$ $1's$ in the former columns. In this case, we have to start some $1's$ in the next columns like

$$\left(\begin{array}{ccccccc|cc} * & \cdots & * & \cdots & * & \cdots & * & 0 & \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots & \\ * & \cdots & 1 & \cdots & 1 & \cdots & * & 0 & \cdots \\ \hline * & \cdots & 0 & \cdots & 0 & \cdots & * & 1 & \cdots \end{array} \right) \begin{array}{c} c_0 \\ \vdots \\ c_{i-1} \\ \hline c_i. \end{array}$$

It is of the form (5.6).

For a monomial $g = x_1 \cdots x_b \cdot y_1 \cdots y_m$ of $(b,m,0)$-type, we may choose l elements in the y part and replace the element $h_{i,j+1}$ by $b_{i,j}$. Then we get a generator $g_1 = x_1 \cdots x_b \cdot y_1 \cdots y_{m-l} z_1 \cdots z_l$ of type $(b, m-l, l)$ with homological dimension $b+m+l$.

Lemma 5.7 *Any monomial of the form $g_1 = x_1 \cdots x_b \cdot y_1 \cdots y_{m-l} z_1 \cdots z_l$ is detected by some $g = x_1 \cdots x_b \cdot y_1 \cdots y_m$ of $(b, m, 0)$-type as above.*

Proof Consider the degree matrix of g_1. In this case, the columns of the $y_{i,j}$ part and of $z_{i,j}$ part are shuffled so that the resulting matrix satisfies condition (4.2) also.

From the discussion above, we get the method to detect all the generators of $E_1^{s,t,b,*}$ as follows:

Method 5.8 For a given (s, t, b) we detect the generators of $E_1^{s,t,b,*}$ as follows:

S1 Express t/q by the p-adic number, and then $t = q(\bar{c}_0 + \bar{c}_1 p + \cdots + \bar{c}_n p^n)$.

S2 List up all the possible carry sequence $\mathbf{k}$ such that in the corresponding sum of row sequence $\mathbf{c}$ the $\widetilde{m}$ determined by (5.1) $\leqslant s - b$.

S3 For each sum of row sequence $\mathbf{c}$, solve the $(b, \widetilde{m}, 0)$-type degree equation by the Simplest way 5.3, from which we get a set S_0 of generators in $F_1^{b+\widetilde{m},t,b,*}$. Notice that the homological dimension of each monomial is $b + \widetilde{m}$ rather than s.

S4 For each $g \in S_0$ replace $(s - b - \widetilde{m})$-factors of $h_{i,j+1}$ by $b_{i,j}$ if possible so that the resulting monomials have homological dimension s. Lemma 5.7 assures that we got all the generators of $(b, \widetilde{m} - (s - b - \widetilde{m}), s - b - \widetilde{m})$-type generators in $F_1^{s,t,b,*}$. Denote the resulting set of generators by G_0

S5 Compute $D_1\{g\}$ for all $g \in S_0$. Then we get monomials of $(b, \widetilde{m}+1, 0)$-type. Lemma 5.5 assures that we got all the generators of $(b, \widetilde{m}+1, 0)$-type. Denote the resulting set of monomials by S_1.

S6 For each $g' \in S_1$ replace $(s - b - \widetilde{m} - 1)$-factors of $h_{i,j+1}$ by $b_{i,j}$ so that the resulting monomials have homological dimension s. Denote the resulting set of generators by G_1.

S7 Repeat S5 and S6 on S_1 until we get $S_k = D_1^k\{S_0\}$ such that $b + \widetilde{m} + k = s$.

S8 Send all the generators in $G_0, \ldots, G_{k-1}$ and S_k to $E_1^{s,t,b,*}$, then we get all the generators of $E_1^{s,t,b,*}$.

6 The Applications

As an application, we introduce a method to prove the convergence of some composition elements in the ASS. We can use the method to check some known results ([cf. [10–18]).

Denote $E[Q_0, \ldots, Q_n]$ by $E(n)$ where Q_i is the dual of τ_i. The Smith–Toda spectrum $V(n)$ characterized by $H^*V(n) = E(n)$ is known to exist for $p > 2n$ and $n \leqslant 3$ (cf. [8]). To assure the existence of $V(3)$, we shall assume the prime $p > 5$ for the remainder of this section.

For $k \leqslant n - 1$, let

$$i_1 : S \longrightarrow V(k) \quad \text{and} \quad i_2 : V(k) \longrightarrow V(n-1)$$

be the inclusion map. Let $(i_1)_* : \mathrm{Ext}_A^{*,*}(Z/p, Z/p) \to \mathrm{Ext}_A^{*,*}(E(k), Z/p)$ be the map induced by i_1. Cohen in [19] proved that $(i_1)_*(h_0 h_n)$ survives to E_∞ in the ASS for the Moore spectrum M. The Moore spectrum M is denoted by $V(0)$ for convenience. From this Cohen also proved in [19] that $h_0 b_{n-1}$ survives to E_∞ in the ASS for the sphere spectrum. Lin [20–22] proved that $(i_1)_*(g_0 h_n)$, $(i_1)_*(g_0 b_{n-1})$ survive to E_∞ in the ASS for $V(1)$, $(i_1)_*(h_1 h_n)$ survives to E_∞ in the ASS for the Moore spectrum. Lin also proved in [20] that $b_0h_n - h_1b_{n-1}$ survives to E_∞ in the ASS for the sphere spectrum.

It is well known that the following composition of maps

$$S^{s\cdot|v_n|} \xrightarrow{i} \Sigma^{s\cdot|v_n|}V(n-1) \xrightarrow{v_n^s} V(n-1) \xrightarrow{j} S^*$$

is referred as the n-th Greek letter elements $\alpha_s^{(n)}$. For $n = 1, 2, 3$ they are the elements α_s, β_s and γ_s respectively. Wang and Zheng in [11] proved that the n-th Greek letter elements $\alpha_s^{(n)}$ is represented in the ASS by

$$\alpha_s^{(n)} = \widetilde{i} \wedge \underbrace{\widetilde{v}_n \wedge \cdots \wedge \widetilde{v}_n}_{s} \wedge \widetilde{j} \in \mathrm{Ext}_A^{s,*}(Z/p, Z/p),$$

where $\wedge$ denotes the Yoneda product and $\widetilde{i} = 1[\ ]1 \in \mathrm{Ext}_A^0(E(n-1), Z/p)$, $\widetilde{j} = Q_0 \cdots Q_{n-1}[\ ]1 \in \mathrm{Ext}_A^0(Z/p, E(n-1))$. Sending $\alpha_s^{(n)}$ to the E_1-term of MSS, we see that

$$\alpha_s^{(n)} = \frac{s!}{(s-n)!} a_n^{s-n} h_{n,0} h_{n-1,1} \cdots h_{1,n-1}$$

and $\frac{s!}{(s-n)!} \not\equiv 0$ for $s \not\equiv 0, 1, \ldots, n-1 \bmod p$.

From the Thom map $\Phi : \mathrm{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*) \to \mathrm{Ext}_A^{*,*}(Z/p, Z/p)$ (cf. [23]), we see that

$$\Phi(\beta_1) = -b_0$$

and

$$\Phi(\gamma_2) = 2b_{2,0}h_{1,2} - 2h_{2,1}b_{1,1}.$$

The cohomology class represented by $2b_{2,0}h_{1,2} - 2h_{2,1}b_{1,1}$ is denoted by $2b_{20}h_2$. Thus the Greek letter elements β_1 and γ_2 are represented by $-b_0$ and $b_{20}h_2$ respectively. Indeed $(i_1)_*(-h_1) \in \mathrm{Ext}_A^{1,*}(E(0), Z/p)$ survives to the following composition of maps:

$$S^{q(1+p)} \longrightarrow \Sigma^{q(p+1)}V(1) \xrightarrow{v_2} V(1) \longrightarrow \Sigma^{q+1}V(0),$$

and the following composition

$$S^{q(1+p)} \to \Sigma^{q(p+1)}V(1) \xrightarrow{v_2} V(1) \longrightarrow \Sigma^{q+1}V(0) \longrightarrow \Sigma^{q+2}S$$

is β_1.

From the discussion above we summarize some of the convergent elements as follows:

$V(2)$	$V(1)$	$V(0) = M$	S
Generator	Generator	Generator	Generator
$(i_1)_*(g_0h_n)$	$(i_1)_*(g_0)$	$(i_1)_*(h_0h_n)$	$\widetilde{\alpha}_s^{(n)}$
$(i_1)_*(g_0b_{n-1})$	$(i_1)_*(h_2)$	$(i_1)_*(h_1h_n)$	b_0
		$(i_1)_*(h_1)$	$2b_{20}h_2$

Suppose that $(i_1)_*(x)$ survives non-trivially to a homotopy element $f : S^* \to V(k)$. For $0 \leqslant k \leqslant n-1$, consider the following composition of maps

$$S^* \xrightarrow{f} V(k) \xrightarrow{i_2} V(n-1) \xrightarrow{v_n^s} \Sigma^{-*}V(n-1) \xrightarrow{j} S.$$

It is easy to see that this composition is represented by the Yoneda product

$$(i_1)_*(x) \wedge \widetilde{i_2} \wedge \underbrace{\widetilde{v_n} \wedge \cdots \wedge \widetilde{v_n}}_{s} \wedge \widetilde{j} = x \wedge \widetilde{i_1} \wedge \widetilde{i_2} \wedge \underbrace{\widetilde{v_n} \wedge \cdots \wedge \widetilde{v_n}}_{s} \wedge \widetilde{j} = x \cdot \widetilde{\alpha}_s^{(n)}.$$

Thus for $s \not\equiv 0, 1, \ldots, n-1$, the element $x \cdot \widetilde{\alpha}_s^{(n)}$ survives non-trivially to a homotopy element of sphere if

1.1 $x \cdot \widetilde{\alpha}_s^{(n)}$ is not zero in the Ext groups $\mathrm{Ext}_A^{*,*}(Z/p, Z/p)$.

1.2 No higher Adams differential hits $x \cdot \widetilde{\alpha}_s^{(n)}$.

From the discussion above one may consider the convergence of the following composition elements and any other elements of the form $x \cdot \widetilde{\alpha}_s^{(n)}$.

Generator	Index	Generator	Index
$h_0 h_n \cdot \widetilde{\beta}_s$	$s < 2p-1$	$h_0 h_n \cdot \widetilde{\gamma}_s$	$s < 2p-1$
		$h_1 h_n \cdot \widetilde{\gamma}_s$	$s < 2p-1$
		$h_1 \cdot \widetilde{\gamma}_s$	$s < 2p-1$
		$g_0 h_n \cdot \widetilde{\gamma}_s$	$s < 2p-1$
		$g_0 b_n \cdot \widetilde{\gamma}_s$	$s < p$

Furthermore, suppose that x converges non-trivially to a homotopy element f of sphere and y converges non-trivially to g. Then the composition element $f \cdot g$ of $\pi_* S$ is represented by $x \cdot y$ in the ASS. The composition $f \cdot g$ is non-trivial if:

2.1 $x \cdot y$ is not zero in the Ext groups.

2.2 No higher Adams differential hits $x \cdot y$ in the ASS.

Indeed, suppose that x and y are elements of the Ext groups and represented by $\bar{x}$ and $\bar{y}$ in the MSS. Then $x \cdot y \neq 0$ if $\bar{x} \cdot \bar{y}$ is not zero in May E_1-term and no higher May differential hits $\bar{x} \cdot \bar{y}$ in MSS. In this case, if $\bar{x} \cdot \bar{y} \in E_1^{s+1,t,b,M}$, we need to compute the corresponding May E_1-term $E_1^{s,t,b,*}$. Furthermore if $x \cdot y \neq 0$ in Adams E_2-term $E_2^{s+1,t+b}$, we could prove that no higher Adams differential hits $x \cdot y$ by showing all the Adams E_2-terms $E_2^{s-r+1,t+b-r+1}$ being zero. In this case we also start from the computation of the May E_1-term $E_1^{s-r+1,t,b-r+1,*}$. Method 5.8 is efficient in detecting the generators of $E_1^{s-r+1,t,b-r+1,*}$.

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