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A NEW FAMILY OF FILTRATION $s+6$ IN THE STABLE HOMOTOPY GROUPS OF SPHERES*

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Abstract In the year 2002, Lin detected a nontrivial family in the stable homotopy groups of spheres $\pi_{t-6}S$ which is represented by $h_n g_0 \tilde{\gamma}_3 \in \mathrm{Ext}_A^{6,t}(Z_p, Z_p)$ in the Adams spectral sequence, where $t = 2p^n(p-1) + 6(p^2+p+1)(p-1)$ and $p \geq 7$ is a prime number. This article generalizes the result and proves the existence of a new nontrivial family of filtration $s+6$ in the stable homotopy groups of spheres $\pi_{t_1-s-6}S$ which is represented by $h_n g_0 \tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+6,t_1}(Z_p, Z_p)$ in the Adams spectral sequence, where $n \geq 4$, $0 \leq s < p-4$, $t_1 = 2p^n(p-1) + 2(p-1)((s+3)p^2 + (s+3)p + (s+3)) + s$.

Key words Stable homotopy groups of spheres, Adams spectral sequence, Toda-Smith spectrum, May spectral sequence

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1 Introduction

The problem of understanding the stable homotopy groups of spheres has long been one of the touchstones of algebraic topology. Low dimensional computation has proceeded slowly and has given little insight into the general structure of π_*S. Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at an odd prime number $p \geq 7$. One of the main tools to determine the stable homotopy groups of spheres π_*S is the Adams spectral sequence (ASS) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \Rightarrow \pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A. If a family of homotopy generators x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of homotopy elements f_i in π_*S and we say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in π_*S have been detected. For example, a family $\zeta_{n-1} \in \pi_{p^nq+q-3}S$ for $n \geq 2$ which has filtration 3 in the ASS and is represented by $h_0 b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p)$ has been detected in reference [1], where $q = 2(p-1)$. In [2] Lin detected a new nontrivial family of homotopy elements in π_*S. The purpose of this article is to generalize and improve his result.

By reference [3] we have that $\mathrm{Ext}_A^{1,*}(Z_p, Z_p)$ has Z_p-bases consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(Z_p, Z_p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ has Z_p-bases consisting of α_2, a_0^2, $a_0h_i(i >$

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0), $g_i(i \geq 0)$, $k_i(i \geq 0)$, $b_i(i \geq 0)$, and $h_ih_j(j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, $p^iq+1, p^{i+1}q+2p^iq$, $2p^{i+1}+p^iq$, $p^{i+1}q$ and p^iq+p^jq respectively.

Let M be the Moore spectrum modulo a prime number $p \geq 5$ given by the the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

Let $\alpha : \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M,$$

where $q = 2(p-1)$. This spectrum which we briefly write as K is known to be the Toda-Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q}K \to K$ given by the cofibration

$$\Sigma^{(p+1)q}K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1}K.$$

We fix $q = 2(p-1)$. In this article, we will prove the following.

Theorem Let $p \geq 7$, $n \geq 4$, then

$$h_n g_0 \tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+6, p^nq+(s+3)p^2q+(s+3)pq+(s+3)q+s}(Z_p, Z_p)$$

is a permanent cycle in the ASS and converges in the Adams spectral sequence to a nontrivial element in $\pi_{p^nq+(s+3)p^2q+(s+3)pq+(s+3)q-6}$, where $0 \leq s < p-4, q = 2(p-1)$.

Note that the $h_n g_0 \tilde{\gamma}_{s+3}$-element obtained in the theorem is an indecomposable element in $\pi_* S$, that is, it is not a composition of elements of low filtration in $\pi_* S$, because $h_n \in \mathrm{Ext}_A^{1,p^nq}(Z_p, Z_p)$, $g_0 \in \mathrm{Ext}_A^{2,pq+2q}(Z_p, Z_p)$ are known to die in the ASS.

After giving some preliminaries on Ext groups of lower dimension in Section 2, the proof of the main theorem will be given in Section 3.

2 Some Preliminaries on Ext Groups

In this section, we will prove some results on Ext groups of lower dimension which will be used in the proof of the main theorem.

From reference [4], there is a May spectral sequence(MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} | m > 0, i \geq 0) \bigotimes P(b_{m,i} | m > 0, i \geq 0) \bigotimes P(a_n | n \geq 0),$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}.$$

One has $d_r : E_r^{s,t,*} \to E_r^{s+1,t,*}$ and if $x \in E_r^{s,t,*}$, $y \in E_r^{s',t',*}$, then $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y)$ $(r \geq 1)$. $xy = (-1)^{ss'+tt'} yx$ for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by $d_1(h_{i,j}) = \sum\limits_{0<k<i} h_{i-k,k+j} h_{k,j}$, $d_1(a_i) = \sum\limits_{0 \leq k < i} h_{i-k,k} a_k$ and $d_1(b_{i,j}) = 0$.

For any element $x \in E_1^{s,t,*}$, define $\dim x = s$ and $\deg x = t$. Then we have:

$$\begin{aligned}
&\dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2, \\
&\deg h_{i,j} = 2(p^i-1)p^j = 2(p-1)(p^{i+j-1} + \cdots + p^j), \\
&\deg b_{i,j} = 2(p^i-1)p^{j+1} = 2(p-1)(p^{i+j} + \cdots + p^{j+1}), \\
&\deg a_i = 2p^i - 1 = 2(p-1)(p^{i-1} + \cdots + 1) + 1, \deg a_0 = 1,
\end{aligned}$$

where $i \geq 1, j \geq 0$.

Proposition 2.1 Let $t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e$ be a positive integer with $0 \leq c_i < p$ $(0 \leq i \leq n)$, $0 \leq e < q$, s be a positive integer with $0 < s < p$. If for some j $(0 \leq j \leq n)$, $s < c_j$, then we have $E_1^{s,t,*} = 0$.

Proof Suppose that $h = x_1 x_2 \cdots x_m$ is the generator of $E_1^{s,t,*}$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. $\deg x_i = q(c_{i,n}p^n + \cdots + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}\deg h &= \sum_{i=1}^{m} \deg x_i = q((\sum_{i=1}^{m} c_{i,n})p^n + \cdots (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})) + (\sum_{i=1}^{m} e_i)\\ &= q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e,\\ \dim h &= \sum_{i=1}^{m} \dim x_i = s.\end{aligned}$$

By the facts that $\dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2$, we know that $0 < m \leq s$. Note that $c_{i,j} = 0$ or 1, $b_i = 0$ or 1 and $m < p$, we have: $\sum_{i=1}^{m} e_i = e$, $\sum_{i=1}^{m} c_{i,0} = c_0$, $\sum_{i=1}^{m} c_{i,1} = c_1$, $\cdots$, $\sum_{i=1}^{m} c_{i,j-1} = c_{j-1}$, $\sum_{i=1}^{m} c_{i,j} = c_j$, $\cdots$, $\sum_{i=1}^{m} c_{i,n} = c_n$.

Note the supposition that $c_{i,j} = 0$ or 1, from the equality $\sum_{i=1}^{m} c_{i,j} = c_j$, we have $m \geq c_j$. But we also know that $c_j > s$ for some j, so $m > s$. Therefore we have $s \geq m > s$. That is impossible. This finishes the proof of Proposition 2.1.

Proposition 2.2 For $0 \leq s < p-3$, the element $\underbrace{a_3 a_3 \cdots a_3}_{s} h_{3,0}h_{2,1}h_{1,2} \in E_1^{s+3,t,*}$ converges to $\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,t}(Z_p, Z_p)$ in the MSS, where $t = (s+3)p^2q + (s+2)pq + (s+1)q + s$ and $\tilde{\gamma}_{s+3}$ converges to $\gamma_{s+3} \in \pi_{(s+3)p^2q+(s+2)pq+(s+1)q-3}S$ (the third Greek letter family element in $\pi_* S$)in the Adams spectral sequence.

Proof See [5].

Proposition 2.3 Let $p \geq 7$, $n \geq 4$, $0 \leq s < p-4$, then

$$h_n g_0 \tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+6, p^n q + (s+3)p^2 q + (s+3)pq + (s+3)q + s}(Z_p, Z_p).$$

Proof First consider the structure of $E_1^{s+5,t,*}$ in the MSS, where $t = p^n q + (s+3)p^2 q + (s+3)pq + (s+3)q + s$. Since $0 \leq s < p-4$, then $5 \leq s+5 < p+1$.

Case 1: $5 \leq s+5 < p$. Suppose that $h = x_1 x_2 \cdots x_m$ is the generator of $E_1^{s+5,t,*}$, where $m \leq s+5$, x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}\deg h &= \sum_{i=1}^{m} \deg x_i = q((\sum_{i=1}^{m} c_{i,n})p^n + \cdots + (\sum_{i=1}^{m} c_{i,2})p^2 + (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})) + (\sum_{i=1}^{m} e_i)\\ &= q(p^n + (s+3)p^2 + (s+3)p + (s+1)) + s.\end{aligned}$$

Note that $c_{i,j} = 0$ or $c_{i,j} = 1$, $e_i = 0$ or 1, $m \leq s+5 < p$, so we have

$$\sum_{i=1}^{m} e_i = s, \quad \sum_{i=1}^{m} c_{i,0} = s+3, \quad \sum_{i=1}^{m} c_{i,1} = s+3,$$

$$\sum_{i=1}^{m} c_{i,2}=s+3,\ \sum_{i=1}^{m} c_{i,3}=\cdots=\sum_{i=1}^{m} c_{i,n-1}=0,\ \sum_{i=1}^{m} c_{i,n}=1.$$

From the above results, it is easy to know that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's. We denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h=x_1x_2\cdots x_{m-1}h_{1,n}$ or $h=x_1x_2\cdots x_{m-1}b_{1,n-1}$.

Subcase 1.1: If $h=x_1x_2\cdots x_{m-1}h_{1,n}$, then $h'=x_1x_2\cdots x_{m-1}\in E_1^{s+4,t-p^nq,*}$ and we have

$$\sum_{i=1}^{m-1} e_i=s,\ \sum_{i=1}^{m-1} c_{i,0}=s+3,\ \sum_{i=1}^{m-1} c_{i,1}=s+3,$$

$$\sum_{i=1}^{m-1} c_{i,2}=s+3,\ \sum_{i=1}^{m} c_{i,3}=\cdots=\sum_{i=1}^{m-1} c_{i,n-1}=0.$$

Note that $c_{i,2}=0$ or 1, we can see that $m-1\geq s+3$ from the equality $\sum\limits_{i=1}^{m-1} c_{i,2}=s+3$, then $m\geq s+4$. Since $m\leq s+5$, then $m=s+4$ or $m=s+5$.

Note the facts that $\sum\limits_{i=1}^{m-1} e_i=s$, $\deg b_{i,j}\equiv 0(\bmod q)$ $(i>0,j\geq 0)$, $\deg h_{i,j}\equiv 0(\bmod q)$ $(i>0,j\geq 0)$ and $\deg a_i\equiv 1(\bmod q)$ $(i\geq 0)$, then the generator h' must have a factor $a_{j_1}a_{j_2}\cdots a_{j_s}$ $(j_1\leq j_2\leq\cdots\leq j_s)$. Note the degrees of a_i's, we can suppose that $h'=\underbrace{a_0\cdots a_0}_{x}\underbrace{a_1\cdots a_1}_{y}\underbrace{a_2\cdots a_2}_{z}\underbrace{a_3\cdots a_3}_{k}x_{s+1}x_{s+2}x_{s+3}\cdots x_{m-1}$, where $0\leq x,y,z,k\leq s$, $x+y+z+k=s$. Then we get that

$$x+y+z+k+\sum_{i=s+1}^{m-1} e_i=s,\ \ y+z+k+\sum_{i=s+1}^{m-1} c_{i,0}=s+3,$$

$$z+k+\sum_{i=s+1}^{m-1} c_{i,1}=s+3,\ \ k+\sum_{i=s+1}^{m-1} c_{i,2}=s+3.$$

If $m=s+4$, we can get that $k=s+3-(\sum\limits_{i=s+1}^{s+3} c_{i,2})\geq s+3-3=s$ from the equality $k+\sum\limits_{i=s+1}^{s+3} c_{i,2}=s+3$. At the same time, we also have that $k\leq s$, so $k=s$, $x=y=z=0$. Then $h'=\underbrace{a_3a_3\cdots a_3}_{s}x_{s+1}x_{s+2}x_{s+3}$ and $x_{s+1}x_{s+2}x_{s+3}\in E_1^{4,q(3p^2+3p+1),*}$. Note that there will exist a element of $b_{i,j}$'s, or otherwise, we would have $\dim x_{s+1}x_{s+2}x_{s+3}=3<4$. That is impossible. On the other hand, there must exist a factor $h_{1,0}h_{2,0}h_{3,0}$ in h' by the facts that $\sum\limits_{i=s+1}^{s+3} c_{i,0}=3$, $\deg h_{i,j}\equiv 0(\bmod pq)$ $(i\geq 1,j>0)$, $\deg h_{i,0}\equiv q(\bmod pq)$ $(i\geq 1)$ and $\deg b_{i,j}\equiv 0(\bmod pq)$ $(i\geq 1,j\geq 0)$. Note the above results, we can easily get that h' is impossible to exist, so h is impossible to exist when $m=s+4$.

If $m=s+5$, we can get that $k=s+3-\sum\limits_{i=s+1}^{s+4} c_{i,2}\geq s+3-4=s-1$ from the equality $k+\sum\limits_{i=s+1}^{s+4} c_{i,2}=s+3$, so $k=s$ or $k=s-1$. When $k=s,x=y=z=0$, then we have

$$\sum_{i=s+1}^{s+4} e_i=0,\ \sum_{i=s+1}^{s+4} c_{i,0}=3,\ \sum_{i=s+1}^{s+4} c_{i,1}=3\quad\text{and}\quad\sum_{i=s+1}^{s+4} c_{i,2}=3.$$

Note that $x_{s+1}x_{s+2}x_{s+3}x_{s+4} \in E_1^{4,*,*}$, we can show that h' is impossible to exist. When $k = s-1, z = 1, x = y = 0$, then we have

$$\sum_{i=s+1}^{s+4} e_i = 0, \sum_{i=s+1}^{s+4} c_{i,0} = 3, \sum_{i=s+1}^{s+4} c_{i,1} = 3 \quad \text{and} \quad \sum_{i=s+1}^{s+4} c_{i,2} = 4.$$

Note that $x_{s+1}x_{s+2}x_{s+3}x_{s+4} \in E_1^{4,*,*}$, it is easy to show that at this time h' is impossible to exist. Therefore at this time h is impossible to exist. Similarly, we can show that when $k = s-1, y = 1, x = y = 0$ and $k = s-1, x = 1, y = z = 0$, h' is impossible to exist either.

From the above discussion, we can get that the generators of the form $h = x_1x_2\cdots x_{m-1}h_{1,n}$ is impossible to exist.

Subcase 1.2: If $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$, $h'' = x_1x_2\cdots x_{m-1} \in E_1^{s+3,t-p^nq,*}$ and we have

$$\sum_{i=1}^{m-1} e_i = s, \sum_{i=1}^{m-1} c_{i,0} = s+3, \sum_{i=1}^{m-1} c_{i,1} = s+3 \quad \text{and} \quad \sum_{i=1}^{m-1} c_{i,2} = s+3.$$

We can get that $m \geq s+4$ from the equality $\sum_{i=1}^{m-1} c_{i,2} = s+3$. But $m \leq s+5$, so $m = s+4$ or $m = s+5$.

When $m = s+5$, $h'' = x_1x_2\cdots x_{s+4} \in E^{s+3,t-p^nq,*}$. But we have $s+3 = \deg h'' \geq s+4$, so when $m = s+5$ h is impossible to exist.

When $m = s+4$, $h'' = x_1x_2\cdots x_{s+3} \in E_1^{s+3,q((s+3)p^2+(s+3)p+(s+1))+s,*}$. By the same method in the proof of Proposition 2.2, we can show $E_1^{s+3,q((s+3)p^2+(s+3)p+(s+3))+s,*} = 0$.

From the above results, we get that the generator of the form $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$ is impossible to exist.

From Subcase 1.1 and Subcase 1.2, we get that when $5 \leq s+5 < p$, $E_1^{s+5,t,*} = 0$. So $E_r^{s+5,t,*} = 0$ and $d_r(E_r^{s+5,t,*}) = 0$. Moreover, we also know that $h_{1,n}, h_{1,0}h_{2,0}, \underbrace{a_3a_3\cdots a_3}_{s}$ $h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$ are permanent cycles in the MSS and converge in the MSS nontrivially to $h_n, g_0, \tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ for $n \geq 0$ respectively (cf.Proposition 2.2), so $h_{1,n}h_{1,0}h_{2,0}\underbrace{a_3a_3\cdots a_3}_{s}$ $h_{3,0}h_{2,1}h_{1,2} \in E_1^{s+6,t,*}$ is a permanent cycle in the MSS. From the above results, we see that the permanent cycle $h_{1,n}h_{1,0}h_{2,0}\underbrace{a_3a_3\cdots a_3}_{s} \in E_1^{s+6,t,*}$ does not bound and converges nontrivially to $h_ng_0\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+6,t}(Z_p, Z_p)$ in the MSS. That is to say, $h_ng_0\tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+6,t}(Z_p, Z_p)$.

Case 2: If $s+5 = p$, then $E_1^{s+5,t,*} = E_1^{p,t',*}$, where $t' = p^nq + (p-2)p^2q + (p-2)pq + (p-2)q + (p-5)$. Suppose that $h = x_1x_2\cdots x_m$ is the generator of $E_1^{p,t',*}$, where $m \leq p$, x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}\deg h &= \sum_{i=1}^{m} \deg x_i \\ &= q((\sum_{i=1}^{m} c_{i,n})p^n + \cdots + (\sum_{i=1}^{m} c_{i,2})p^2 + (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})) + (\sum_{i=1}^{m} e_i) \\ &= q(p^n + (p-2)p^2 + (p-2)p + (p-2)) + p - 5.\end{aligned}$$

By the knowledge about p-adic expression in number theory and the suppositions that $m \leq p$, $c_{i,j} = 0$ or 1, $e_i = 0$ or 1, we have

$$\sum_{i=1}^{m} e_i = p-5, \ \sum_{i=1}^{m} c_{i,0} = p-2, \ \sum_{i=1}^{m} c_{i,1} = p-2,$$

$$\sum_{i=1}^{m} c_{i,2} = p-2, \ (\sum_{i=1}^{m} c_{i,3})p^3 + \cdots + (\sum_{i=1}^{m} c_{i,n})p^n = p^n.$$

From the above equality, we have that $(\sum\limits_{i=1}^{m} c_{i,3}) + \cdots + (\sum\limits_{i=1}^{m} c_{i,n})p^{n-3} = p^{n-3}$. Therefore, $p|\sum\limits_{i=1}^{m} c_{i,3}$. Note that $c_{i,3} = 0$ or 1, $m \leq p$, it is easy to know that $\sum\limits_{i=1}^{m} c_{i,3} = 0$ or $\sum\limits_{i=1}^{m} c_{i,3} = p$.

Subcase 2.1: If $\sum\limits_{i=1}^{m} c_{i,3} = 0$ and $n = 4$, it is easy to get that $\sum\limits_{i=1}^{m} c_{i,4} = 1$, so there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's.

If $n > 4$, then $(\sum\limits_{i=1}^{m} c_{i,4})p^4 + \cdots + (\sum\limits_{i=1}^{m} c_{i,n})p^n = p^n$, so $(\sum\limits_{i=1}^{m} c_{i,4}) + (\sum\limits_{i=1}^{m} c_{i,5})p \cdots + (\sum\limits_{i=1}^{m} c_{i,n})p^{n-4}$ $= p^{n-4}$. Similarly we know that $\sum\limits_{i=1}^{m} c_{i,4} = 0$ or $\sum\limits_{i=1}^{m} c_{i,4} = p$.

We claim that if $\sum\limits_{i=1}^{m} c_{i,3} = 0$, then $\sum\limits_{i=1}^{m} c_{i,4} = 0$, for otherwise, we would have $\sum\limits_{i=1}^{m} c_{i,4} = p$, then $m = p$. Since $\dim h = \sum\limits_{i=1}^{m} \dim x_i = p$, then for any $1 \leq i \leq p$, $\dim x_i = 1$. So we get that $h \in P(a_n|n \geq 0) \otimes E(h_{m,i}|m > 0, i \geq 0)$. For any $1 \leq i \leq m = p$, $\deg x_i =$ higher terms $+p^4q+$lower terms. Since $\sum\limits_{i=1}^{p} e_i = p-5$, $\deg a_i \equiv 1(\mathrm{mod} q)$ $(i \geq 0)$ and $\deg h_{i,j} \equiv 0(\mathrm{mod} q)$ $(i > 0, j \geq 0)$, then there exists a factor $a_{j_1}a_{j_2}\cdots a_{j_{p-5}}$ $(0 \leq j_1 \leq j_2 \leq \cdots \leq j_{p-5} \leq n+1)$ among x_i's such that for any $1 \leq i \leq p-5, j_i \geq 5$ and $\deg a_{j_i}$ =higher terms $+p^4q+p^3q+p^2q+pq+q+1$. It is obvious that $\sum\limits_{i=1}^{m} c_{i,3} \geq p-5$ which contradicts $\sum\limits_{i=1}^{m} c_{i,3} = 0$, thus the claim follows.

By induction on j we can get that $\sum\limits_{i=1}^{m} c_{i,j} = 0 (4 \leq j \leq n-1)$, so $\sum\limits_{i=1}^{m} c_{i,n} = 1$, that is to say, there is a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's.

In all, at this time for $n \geq 4$, there is a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's $(n \geq 4)$. We denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h = x_1x_2\cdots x_{m-1}h_{1,n}$ or $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$.

If $h = x_1x_2\cdots x_{m-1}h_{1,n}$, then $h' = x_1x_2\cdots x_{m-1} \in E_1^{p-1,t'-p^nq,*}$. By the same discussion in Subcase 1.1, we can easily show that h' is impossible to exist. Thus at this time h' is impossible to exist.

If $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$, then $h'' = x_1x_2x_3\cdots x_{m-1} \in E_1^{p-2,t'-p^nq,*}$. By the same method in the proof of Proposition 2.2, we can get that $E_1^{p-2,t'-p^nq,*} = 0$, so $h'' = x_1x_2x_3\cdots x_{m-1}$ is impossible to exist and $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$ is also impossible to exist.

Subcase 2.2: If $\sum\limits_{i=1}^{m} c_{i,3} = p$, then $m = p$. Since $\dim h = p$, we can easily see that there cannot be $b_{i,j}$'s among x_i's and $h = x_1x_2\cdots x_p \in E(h_{m,i}|m > 0, i \geq 0) \bigotimes P(a_n|n \geq 0)$. For $n = 4$, we can easily get that $\sum\limits_{i=1}^{p} c_{i,3} = p$, $\sum\limits_{i=1}^{p} c_{i,4} = \cdots = \sum\limits_{i=1}^{p} c_{i,n} = 0$. For $n > 4$, from the equality $(\sum\limits_{i=1}^{p} c_{i,3})p^3 + \cdots + (\sum\limits_{i=1}^{p} c_{i,n})p^n = p^n$, we can have $(\sum\limits_{i=1}^{p} c_{i,4} + 1) + (\sum\limits_{i=1}^{p} c_{i,5})p + \cdots + (\sum\limits_{i=1}^{p} c_{i,n})p^{n-4} =$

p^{n-4}. Then $p|(\sum_{i=1}^{p} c_{i,4}+1)$. Note that $c_{i,4}=0$ or 1, then it is easy to get that $\sum_{i=1}^{p} c_{i,4}=p-1$. By induction on j, we can prove that $\sum_{i=1}^{p} c_{i,j}=p-1$ $(4\le j\le n-1)$. So $\sum_{i=1}^{p} c_{i,n}=0$.

When $n=4$, by the facts that $\sum_{i=1}^{p} e_i=p-5$, $\sum_{i=1}^{p} c_{i,0}=p-2$, $\sum_{i=1}^{p} c_{i,1}=p-2$, $\sum_{i=1}^{p} c_{i,2}=p-2$, $\sum_{i=1}^{p} c_{i,3}=p$, we can prove that $h=x_1x_2\cdots x_p$ is impossible to exist by the same method in the proof of Proposition 2.2.

When $n>4$, by the facts that $\sum_{i=1}^{p} c_{i,3}=p$, $\sum_{i=1}^{p} c_{i,4}=\cdots=\sum_{i=1}^{p} c_{i,n-1}=p-1$, $\deg h_{k,j}=q(p^{k+j-1}+\cdots+p^j)$ $(k\ge 1, j\ge 0)$ and $\deg a_i=q(p^{i-1}+\cdots+p+1)+1$ $(i>0)$, we can divide the p x_i's into two disjoint classes S_1 and S_2. The two disjoint classes are given by

$$S_1=\{x|\deg x=q(p^3+\text{lower terms}\},$$
$$S_2=\{x|\deg x=q(p^{n-1}+\cdots+p^3+\text{lower terms})\}.$$

For a class S, define the number of elements in S by $N(S)$, then we can get $N(S_1)=1$ and $N(S_2)=p-1$. Similarly, by the facts that $\sum_{i=1}^{p} e_i=p-5$, $\sum_{i=1}^{p} c_{i,0}=p-2$, $\sum_{i=1}^{p} c_{i,1}=p-2$, $\sum_{i=1}^{p} c_{i,2}=p-2$, $\sum_{i=1}^{p} c_{i,3}=p$, $\deg h_{k,j}=q(p^{k+j-1}+\cdots+p^j)$ $(k\ge 1, j\ge 0)$ and $\deg a_i=q(p^{i-1}+\cdots+p+1)+1$ $(i>0)$, we can also divide the p x_i's into three disjoint classes. The three classes are given by

$$S_3=\{x|\deg x=q(\text{higher terms}+p^3)\}, N(S_3)=2,$$
$$S_4=\{x|\deg x=q(\text{higher terms}+p^3+p^2+p+1)\}, N(S_4)=3,$$
$$S_5=\{x|\deg x=q(\text{higher terms}+p^3+p^2+p+1)+1\}, N(S_5)=p-5.$$

Since $S_1\bigcup S_2=S_3\bigcup S_4\bigcup S_5$, we have that $S_3\subset S_2$ or $S_3\not\subset S_2$. If $S_3\subset S_2$, then there will be two $h_{n-3,3}$'s such that $\deg h_{n-3,3}=q(p^{n-1}+\cdots+p^3)$. This is impossible since $h_{n-3,3}^2=0$. If $S_3\not\subset S_2$, then $S_4\subset S_2$. Thus there exist three $h_{n,0}$'s among x_i's such that $\deg h_{n,0}=q(p^{n-1}+\cdots+1)$. But this is impossible since $h_{n,0}^2=0$. This shows that $E_1^{p,t',*}=0$ for $n>4$.

From the above discussion, we get that when $s+5=p$, $E_1^{p,t',*}=0$ $(n\ge 4)$. So $E_r^{p,t',*}=0$ and $d_r(E_r^{p,t',*})=0$.

Moreover, we know that $h_{1,n}, h_{1,0}h_{2,0}, \underbrace{a_3a_3\cdots a_3}_{s} h_{3,0}h_{2,1}h_{1,2}\in E_1^{*,*,*}$ are permanent cycles in the MSS and converge in the MSS to nontrivially $h_n, g_0, \tilde{\gamma}_{s+3}\in \mathrm{Ext}_A^{*,*}(Z_p,Z_p)$ for $n\ge 0$ respectively, so $h_{1,n}h_{1,0}h_{2,0}\underbrace{a_3a_3\cdots a_3}_{p-5} h_{3,0}h_{2,1}h_{1,2}\in E_1^{*,*,*}$ is a permanent cycle and converges nontrivially to $h_ng_0\tilde{\gamma}_{p-2}\in \mathrm{Ext}_A^{s+6,t}(Z_p,Z_p)$. That is, $h_ng_0\tilde{\gamma}_{p-2}\ne 0\in \mathrm{Ext}_A^{p+1,t'}(Z_p,Z_p)$.

From Case 1 and Case 2, the proposition follows.

Proposition 2.4 Let $p\ge 7$, $n\ge 4$, $0\le s<p-4$, $2\le r\le s+6$, then in the ASS, $h_ng_0\tilde{\gamma}_{s+3}\ne d_r(x)$ for any $x\in \mathrm{Ext}_A^{s+6-r,q(p^n+(s+3)p^2+(s+3)p+(s+3))+(s-r+1)}(Z_p,Z_p)$.

Proof We only need to prove that $\mathrm{Ext}_A^{s+6-r,t''}(Z_p,Z_p)=0$, where $t''=q(p^n+(s+3)p^2+(s+3)p+(s+3))+(s-r+1)$. Suppose that $h=x_1x_2\cdots x_m$ is the generator of $E_1^{s+6-r,t'',*}$, where $m\le s+6-r$, x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0\le k\le n+1, 0\le l+j\le n+1, 0\le u+z\le$

$n,l > 0, j \geq 0, u > 0, z \geq 0$. $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}\deg h &= \sum_{i=1}^{m} \deg x_i \\ &= q((\sum_{i=1}^{m} c_{i,n})p^n + \cdots + (\sum_{i=1}^{m} c_{i,2})p^2 + (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})) + (\sum_{i=1}^{m} e_i) \\ &= q(p^n + (s+3)p^2 + (s+3)p + (s+3)) + s - r + 1.\end{aligned}$$

We claim that $s-r+1 \geq 0$, or otherwise, we would have $p > \sum_{i=1}^{m} e_i = q + (s-r+1) \geq q-5 \geq p$. That is impossible. The claim follows.

Note the suppositions that $c_{i,j} = 0$ or 1, $e_i = 0$ or 1 and $m \leq s+6-r \leq s+6-2 = s+4 < p$, then we have that $\sum_{i=1}^{m} e_i = s-r+1$, $\sum_{i=1}^{m} c_{i,0} = s+3$, $\sum_{i=1}^{m} c_{i,1} = s+3$, $\sum_{i=1}^{m} c_{i,2} = s+3$, $\sum_{i=1}^{m} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0$ and $\sum_{i=1}^{m} c_{i,n} = 1$.

It is easy to see that there exists a $h_{1,n}$ or $b_{1,n-1}$ among x_i's. We denote $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h = x_1x_2\cdots x_{m-1}h_{1,n}$ or $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$.

Case 1: If $h = x_1x_2\cdots x_{m-1}h_{1,n}$, then $h' = x_1x_2\cdots x_{m-1} \in E_1^{s+5-r,t''-p^nq,*}$ and we have

$$\sum_{i=1}^{m-1} e_i = s-r+1, \quad \sum_{i=1}^{m-1} c_{i,0} = s+3, \quad \sum_{i=1}^{m-1} c_{i,1} = s+3,$$

$$\sum_{i=1}^{m-1} c_{i,2} = s+3, \quad \sum_{i=1}^{m-1} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, \quad \sum_{i=1}^{m-1} c_{i,n} = 0.$$

When $r > 2$, we have that $s+5-r \leq s+5-3 = s+2 < s+3$, then we see that $E_1^{s+5-r,t''-p^nq,*} = 0$ by Proposition 2.1. Thus at this time the generator h is impossible to exist.

When $r = 2$, from the equality $\sum_{i=1}^{m-1} c_{i,2} = s+3$ we have that $m-1 \geq s+3$, so $m \geq s+4$. But we also know $m \leq s+6-2 = s+4$ when $r = 2$. Therefore $m = s+4$. Then $h' = x_1x_2\cdots x_{s+3} \in E_1^{s+3,t''-p^nq,*}$. By the same method in the proof of Proposition 2.2, we can show that $E_1^{s+3,t''-p^nq,*} = 0$.

Case 2: If $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$, then $h'' = x_1x_2\cdots x_{m-1} \in E_1^{s+4-r,t''-p^nq,*}$ and we have

$$\sum_{i=1}^{m-1} e_i = s-r+1, \quad \sum_{i=1}^{m-1} c_{i,0} = s+3, \quad \sum_{i=1}^{m-1} c_{i,1} = s+3,$$

$$\sum_{i=1}^{m-1} c_{i,2} = s+3, \quad \sum_{i=1}^{m-1} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, \quad \sum_{i=1}^{m-1} c_{i,n} = 0.$$

From the equality $\sum_{i=1}^{m-1} c_{i,2} = s+3$ and $s+4-r \leq s+4-2 = s+2$, we can know that at this time $E_1^{s+4-r,t''-p^nq,*} = 0$ by Proposition 2.1.

From Case 1 and Case 2, we see that $E_1^{s+6-r,t'',*} = 0$, so $\mathrm{Ext}_A^{s+6-r,t''}(Z_p, Z_p) = 0$. This finishes the proof of Proposition 2.4.

3 Proof of the Theorem

From Theorem II of [2], we get that $(i'i)_*(h_n g_0) \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(H^*K, Z_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\xi' \in \pi_{p^nq+pq+2q-3}K$. Let $\gamma : \Sigma^{q(p^2+p+1)}V(2) \to V(2)$ be the v_3-map and consider the following composition

$$\tilde{f} : \Sigma^{p^nq+pq+2q-3}S \xrightarrow{\xi'} K \xrightarrow{jj'\bar{j}\gamma^{s+3}\bar{i}} \Sigma^{-(s+3)p^2q-(s+2)pq-(s+1)q+3}S.$$

Since ξ' is represented by $(i'i)_*(h_n g_0) \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(H^*K, Z_p)$, then the above $\tilde{f}$ is represented by

$$\bar{c} = (jj'\bar{j}\gamma^{s+3}\bar{i}i'i)_*(h_n g_0) \in \mathrm{Ext}_A^{s+6,p^nq+(s+3)p^2q+(s+3)pq+(s+3)q+s}(Z_p, Z_p).$$

From Proposition 2.2 and the knowledge of Yoneda products we know that the composition

$$\mathrm{Ext}_A^{0,0}(Z_p, Z_p) \xrightarrow{(\bar{i}i'i)_*} \mathrm{Ext}_A^{0,0}(H^*V(2), Z_p) \xrightarrow{(\gamma_*)^{s+3}(jj'\bar{j})_*} \mathrm{Ext}_A^{s+3,(s+3)p^2q+(s+2)pq+(s+1)q+s}(Z_p, Z_p)$$

is a multiplication (up to nonzero scalar) by

$$\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,(s+3)p^2q+(s+2)pq+(s+1)q+s}(Z_p, Z_p).$$

Hence, $\tilde{f}$ is represented (up to nonzero scalar) by

$$\bar{c} = h_n g_0 \tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+6,p^nq+(s+3)p^2q+(s+3)pq+(s+3)q+s}(Z_p, Z_p)$$

in the ASS (cf. Proposition 2.3).

From Proposition 2.4, we know that $h_n g_0 \tilde{\gamma}_{s+3}$ cannot be hit by the differentials in the ASS and so the corresponding homotopy element $\tilde{f} \in \pi_* S$ is nontrivial and of order p. This finishes the proof of the theorem.

Remark For the above Theorem, if we take $s = 0$, then Theorem I of Reference [2] will be obtained. The theorem we obtained in this article generalizes and improves Theorem I of Reference [2].

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