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A Nontrivial Product of Filtration $s+5$ in the Stable Homotopy of Spheres

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Abstract In this paper, some groups $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ with specialized s and t are first computed by the May spectral sequence. Then we make use of the Adams spectral sequence to prove the existence of a new nontrivial family of filtration $s+5$ in the stable homotopy groups of spheres $\pi_{p^nq+(s+3)pq+(s+1)q-5}S$ which is represented (up to a nonzero scalar) by $\tilde{\beta}_{s+2}b_0h_n \in \mathrm{Ext}_A^{s+5,p^nq+(s+3)pq+(s+1)q+s}(Z_p, Z_p)$ in the Adams spectral sequence, where $p \geq 5$ is a prime number, $n \geq 3$, $0 \leq s < p-3$, $q = 2(p-1)$.

Keywords Stable homotopy of spheres, Adams spectral sequence, Toda–Smith spectrum, May spectral sequence

MR(2000) Subject Classification 55Q45, 55T15

1 Introduction

Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at an odd prime number p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \Rightarrow \pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A. If a family of homotopy generators x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of homotopy elements f_i in π_*S and we say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in π_*S have been detected. For example, a family $\zeta_{n-1} \in \pi_{p^nq+q-3}S$ for $n \geq 2$ which has filtration 3 in the ASS and is represented by $h_0b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p)$ has been detected in reference [1], where $q = 2(p-1)$. In this paper, we detect a family of homotopy elements in π_*S which has filtration $s+5$ in the ASS.

From reference [2], $\mathrm{Ext}_A^{1,*}(Z_p, Z_p)$ has Z_p-bases consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(Z_p, Z_p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ has Z_p-bases consisting of α_2, a_0^2, $a_0h_i(i>0)$, $g_i(i \geq 0)$, $k_i(i \geq 0)$, $b_i(i \geq 0)$, and $h_ih_j(j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, $p^iq+1, p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq, respectively.

Let M be the Moore spectrum modulo a prime number $p \geq 3$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

Let $\alpha : \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1}M,$$

where $q = 2(p-1)$. This spectrum which we write in brief as K is known to be the Toda–Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q}K \to K$ given by the cofibration

$$\Sigma^{(p+1)q}K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1}K.$$

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As we know, in the classical Adams spectral sequence the β-element $\beta_t = jj'\beta^t i'i$ is a nontrivial element of order p in $\pi_{(p+1)tq-q-2}S$, where $p \geq 5$.

In this paper, we will prove the following theorem.

Theorem 1.1 *Let $p \geq 5, n \geq 3$. Then $\tilde{\beta}_{s+2}b_0h_n \neq 0 \in \mathrm{Ext}_A^{s+5,p^nq+(s+3)pq+(s+1)q+s}(Z_p, Z_p)$ is a permanent cycle in the Adams Spectral Sequence and converges to a nontrivial element in $\pi_{p^nq+(s+3)pq+(s+1)q-5}$, where $0 \leq s < p-3$, $q = 2(p-1)$.*

Remark The $\tilde{\beta}_{s+2}b_0h_n$-element obtained in Theorem 1.1 actually is the product of the β-element $\beta_{s+2} = jj'\beta^{s+2}i'i \in \pi_{(s+2)(p+1)q-q-2}S$ and the $(b_0h_n + h_1b_{n-1})$-element $j\xi_n \in \pi_{p^nq+pq-3}S$ in reference [3].

After giving some preliminaries on Ext groups of lower dimension in Section 2, the proof of Theorem 1.1 will be given in Section 3.

2 Some Preliminaries on Ext groups

In this section, we will first prove some results on Ext groups of lower dimension which will be used in the proof of the main theorem.

From [4, Theorem 3.2.5], there is a May spectral sequence (MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i}|m > 0, i \geq 0) \otimes P(b_{m,i}|m > 0, i \geq 0) \otimes P(a_n|n \geq 0),$$

where E is the exterior algebra, P is the polynomial algebra, and $h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}$, $b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}$, $a_n \in E_1^{1,2p^n-1,2n+1}$. One has $d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}$ and if $x \in E_r^{s,t,*}$, $y \in E_r^{s',t',*}$, then $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y)$. $xy = (-1)^{ss'+tt'}yx$ for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j}, \qquad d_1(a_i) = \sum_{0\leq k<i} h_{i-k,k}a_k, \qquad d_1(b_{i,j}) = 0.$$

For any element $x \in E_1^{s,t,*}$, define dim $x = s$, deg $x = t$. Then we have

$$\begin{aligned}
&\dim h_{i,j} = \dim a_i = 1, \quad \dim b_{i,j} = 2,\\
&\deg h_{i,j} = 2(p^i-1)p^j = 2(p-1)(p^{i+j-1} + \cdots + p^j),\\
&\deg b_{i,j} = 2(p^i-1)p^{j+1} = 2(p-1)(p^{i+j} + \cdots + p^{j+1}),\\
&\deg a_i = 2p^i - 1 = 2(p-1)(p^{i-1} + \cdots + 1) + 1,\\
&\deg a_0 = 1,
\end{aligned}$$

where $i \geq 1, j \geq 0$.

Lemma 2.1 *Let $t = q(c_np^n + c_{n-1}p^{n-1} + \cdots + c_1p + c_0) + e$ be a positive integer with $0 \leq c_i < p$ $(0 \leq i \leq n)$, $0 \leq e < q$, s be a positive integer with $0 < s < p$. If for some j $(0 \leq j \leq n)$, $s < c_j$, then in the MSS we have $E_1^{s,t,*} = 0$.*

Proof Suppose that $h = x_1x_2\cdots x_m$ is the generator of $E_1^{s,t,*}$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Assume that deg $x_i = q(a_{i,n}p^n + \cdots + a_{i,1}p + a_{i,0}) + e_i$, where $a_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}
\deg h &= \sum_{i=1}^m \deg x_i\\
&= q\bigg(\Big(\sum_{i=1}^m a_{i,n}\Big)p^n + \cdots \Big(\sum_{i=1}^m a_{i,1}\Big)p + \Big(\sum_{i=1}^m a_{i,0}\Big)\bigg) + \Big(\sum_{i=1}^m e_i\Big)\\
&= q(c_np^n + c_{n-1}p^{n-1} + \cdots + c_1p + c_0) + e,
\end{aligned}$$

$$\dim\, h = \sum_{i=1}^{m} \dim\, x_i = s.$$

By the facts that $\dim\, h_{i,j} = \dim\, a_i = 1$ and $\dim\, b_{i,j} = 2$, we know that $0 < m \le s < p$ from the equality $\sum_{i=1}^{m} \dim\, x_i = s$. Noting that $a_{i,j} = 0$ or 1, $e_i = 0$ or 1 and $m < p$, we have: $\sum_{i=1}^{m} e_i = e$, $\sum_{i=1}^{m} a_{i,0} = c_0$, $\sum_{i=1}^{m} a_{i,1} = c_1, \ldots, \sum_{i=1}^{m} a_{i,j-1} = c_{j-1}$, $\sum_{i=1}^{m} a_{i,j} = c_j, \ldots$, $\sum_{i=1}^{m} a_{i,n} = c_n$. Noting the supposition that $a_{i,j} = 0$ or 1, from the equality $\sum_{i=1}^{m} a_{i,j} = c_j$, we have $m \ge c_j$. But we also know that $c_j > s$, so $m > s$. Therefore we have $s \ge m > s$. That is impossible. This finishes the proof of Lemma 2.1.

Theorem 2.2 *For $p \ge 5$, $0 \le s < p-2$, the element*

$$\underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0} h_{1,1} \in E_r^{s+2,(s+2)pq+(s+1)q+s,*}$$

in the May spectral sequence converges to the second Greek letter family element $\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,t}(Z_p, Z_p)$, where $r \ge 1$, $t = (s+2)pq + (s+1)q + s$, and $\tilde{\beta}_{s+2}$ converges to the β-element $\beta_{s+2} \in \pi_{(s+2)pq+(s+1)q-2}S$ in the Adams spectral sequence.

Proof From [5, Theorems 1 and 2], we know that the β-element $\beta_{s+2} \in \pi_{(s+2)pq+(s+1)q-2}S$ is represented by the second Greek letter family element $\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,t}(Z_p,\, Z_p)$ in the ASS, where $t = (s+2)pq + (s+1)q + s$. However in the MSS, $E_1^{s+2,(s+2)pq+(s+1)q+s,*} = Z_p\{\underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0} h_{1,1}\}$ (This will be proved later.), so in the MSS, $\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,t}(Z_p, Z_p)$ is represented by $\underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0} h_{1,1} \in E_1^{s+2,t,*}$.

Now our remaining work is to prove $E_1^{s+2,t,*} = Z_p\{\underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0} h_{1,1}\}$. Suppose that $h = x_1 x_2 \cdots x_m$ is the generator of $E_1^{s+2,t,*}$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \le k \le 2$, $0 \le l+j \le 2$, $0 \le u+z \le 1$, $l > 0$, $j \ge 0$, $u > 0$, $z \ge 0$. Assume that $\deg x_i = q(a_{i,1}p + a_{i,0}) + e_i$, where $a_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}\deg\, h &= \sum_{i=1}^{m} \deg\, x_i \\ &= q\Big(\Big(\sum_{i=1}^{m} a_{i,1}\Big)p + \Big(\sum_{i=1}^{m} a_{i,0}\Big)\Big) + \Big(\sum_{i=1}^{m} e_i\Big) \\ &= q((s+2)p + (s+1)) + s, \\ \dim\, h &= \sum_{i=1}^{m} \dim\, x_i = s+2.\end{aligned}$$

By the facts that $\dim\, h_{i,j} = \dim\, a_i = 1$ and $\dim\, b_{i,j} = 2$, we know that $0 < m \le s+2$ from $\sum_{i=1}^{m} \dim\, x_i = s+2$. Noting that $a_{i,j} = 0$ or 1, $e_i = 0$ or 1 and $m \le s+2 < p$, we have: $\sum_{i=1}^{m} e_i = s$, $\sum_{i=1}^{m} a_{i,0} = s+1$ and $\sum_{i=1}^{m} a_{i,1} = s+2$. From the equality $\sum_{i=1}^{m} a_{i,1} = s+2$ and the fact that $a_{i,2} = 0$, or $a_{i,2} = 1$, we see that $m \ge s+2$, so $m = s+2$. Since $\dim\, h = \sum_{i=1}^{s+2} \dim\, x_i = s+2$, then for any $1 \le i \le s+2$, $\dim\, x_i = 1$, so we get that $h \in P(a_n | n \ge 0) \bigotimes E(h_{m,i} | m > 0, i \ge 0)$.

Since $\sum_{i=1}^{s+2} e_i = s \equiv s \,(\mathrm{mod}\; q)$, $\deg\, h_{i,j} \equiv 0 \,(\mathrm{mod}\; q)$ $(i > 0, j \ge 0)$ and $\deg\, a_i \equiv 1 \,(\mathrm{mod}\; q)$ $(i \ge 0)$, then the generator h must have a factor $a_{j_1} a_{j_2} \cdots a_{j_s}$. Noting the degrees of a_i's and the commutativity of $E_1^{*,*,*}$, we can suppose that $h = \underbrace{a_0 \cdots a_0}_{x} \underbrace{a_1 \cdots a_1}_{y} \underbrace{a_2 \cdots a_2}_{z} x_{s+1} x_{s+2}$ (up to sign), where $0 \le x, y, z \le s, x+y+z = s$. Then we get that $x + y + z + \sum_{i=s+1}^{s+2} e_i = s$, $y + z + \sum_{i=s+1}^{s+2} a_{i,0} = s+1$ and $z + \sum_{i=s+1}^{s+2} a_{i,1} = s+2$. From the equality $z + \sum_{i=s+1}^{s+2} a_{i,1} = s+2$, we can get that $z = s + 2 - \sum_{i=s+1}^{s+2} a_{i,1} \geq s + 2 - 2 = s$. So $z = s, x = y = 0$, that is, $h = \underbrace{a_2 a_2 \cdots a_2}_{s} x_{s+1} x_{s+2}$. It is easy to show that $x_{s+1}x_{s+2} \in E_1^{2,2pq+q,*} \cong Z_p\{h_{2,0}h_{1,1}\}$. It follows that $h = \underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0}h_{1,1}$ (up to sign) and $E_1^{s+2,t,*} = Z_p\{\underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0}h_{1,1}\}$.

Proposition 2.3 *Let $p \geq 5$, $n \geq 3$, $0 \leq s < p-3$. Then*

$$\tilde{\beta}_{s+2} b_0 h_n \neq 0 \in \mathrm{Ext}_A^{s+5, p^n q + (s+3)pq + (s+1)q + s}(Z_p, Z_p).$$

Proof First consider the structure of $E_1^{s+4,t',*}$ in the MSS, where $t' = p^n q + (s+3)pq + (s+1)q + s$. Since $0 \leq s < p-3$, then $4 \leq s+4 < p+1$. Suppose that $h = x_1 x_2 \cdots x_m$ is the generator of $E_1^{s+4,t',*}$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. deg $x_i = q(a_{i,n}p^n + a_{i,n-1}p^{n-1} + \cdots + a_{i,0}) + e_i$, where $a_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}
\deg h &= \sum_{i=1}^{m} \deg x_i \\
&= q\Big(\Big(\sum_{i=1}^{m} a_{i,n}\Big)p^n + \cdots + \Big(\sum_{i=1}^{m} a_{i,2}\Big)p^2 + \Big(\sum_{i=1}^{m} a_{i,1}\Big)p + \Big(\sum_{i=1}^{m} a_{i,0}\Big)\Big) + \Big(\sum_{i=1}^{m} e_i\Big) \\
&= q(p^n + (s+3)p + (s+1)) + s, \\
\dim h &= \sum_{i=1}^{m} \dim x_i = s+4.
\end{aligned}$$

Noting that dim $x_i = 1$ or 2, we have $m \leq s+4 \leq p$ from $\sum_{i=1}^{m} \dim x_i = s+4$. By the knowledge about the p-adic expression in number theory and the suppositions that $a_{i,j} = 0$ or $a_{i,j} = 1$, $e_i = 0$ or 1, $m \leq s+4 \leq p$, we have

$$\begin{aligned}
&\sum_{i=1}^{m} e_i = s, \qquad \sum_{i=1}^{m} a_{i,0} = s+1, \\
&\sum_{i=1}^{m} a_{i,1} = s+3, \qquad \Big(\sum_{i=1}^{m} a_{i,2}\Big)p^2 + \Big(\sum_{i=1}^{m} a_{i,3}\Big)p^3 + \cdots + \Big(\sum_{i=1}^{m} a_{i,n}\Big)p^n = p^n.
\end{aligned} \tag{$\clubsuit$}$$

Case 1 $0 \leq s < p-4$.

By the knowledge about the p-adic expression in number theory and ($\clubsuit$), we have

$$\begin{aligned}
&\sum_{i=1}^{m} e_i = s, \qquad \sum_{i=1}^{m} a_{i,0} = s+1, \qquad \sum_{i=1}^{m} a_{i,1} = s+3, \\
&\sum_{i=1}^{m} a_{i,2} = \cdots = \sum_{i=1}^{m} a_{i,n-1} = 0, \qquad \sum_{i=1}^{m} a_{i,n} = 1.
\end{aligned}$$

It easy to see that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's. By the commutativity of $E_1^{*,*,*}$, we can denote $h_{1,n}$ or $b_{1,n-1}$ by x_m.

If $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$, $h' = x_1 x_2 \cdots x_{m-1} \in E_1^{s+3,t'-p^n q,*}$. By an argument similar to that used in the proof of Theorem 2.2, we can show that $E_1^{s+3,t'-p^n q,*} = 0$, so h' is impossible to exist. Thus h is impossible to be of the form $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$.

If $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$, $h'' = x_1 x_2 x_3 \cdots x_{m-1} \in E_1^{s+2,t'-p^n q,*}$. By Lemma 2.1, we can know that $E_1^{p-2,t'-p^n q,*} = 0$, so $h'' = x_1 x_2 x_3 \cdots x_{m-1}$ is impossible to exist and h is impossible to be of the form $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$.

From the above discussion, we see that when $0 \leq s < p-4 = 0$, $E_1^{s+4,t',*} = 0$. Thus $E_r^{s+4,t',*} = 0$ for $r \geq 1$. It is known that $h_{1,n}$, $b_{1,n}$, $\underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0}h_{1,1} \in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to h_n, b_n, $\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ for $n \geq 0$, respectively (cf. Theorem 2.2), so at this time the permanent cycle $\underbrace{a_2 a_2 \cdots a_2}_{s} h_{2,0} h_{1,1} b_{1,0} h_{1,n} \in E_r^{s+5,t',*}$ is not bounded and converges nontrivially to $\tilde{\beta}_{s+2} b_0 h_n \in \mathrm{Ext}_A^{s+5,t'}(Z_p, Z_p)$ in the MSS. Thus $\tilde{\beta}_{s+2} b_0 h_n \neq 0 \in \mathrm{Ext}_A^{s+5,t'}(Z_p, Z_p)$.

Case 2 $\quad s = p - 4$

$E_1^{s+4,t',*} = E_1^{p,t'',*}$, where $t'' = p^n q + (p-1)pq + (p-3)q + (p-4)$. Noting that $m \leq s+4 = p$, from (♣) we have

$$\Big(\sum_{i=1}^{m} a_{i,2}\Big)p^2 + \Big(\sum_{i=1}^{m} a_{i,3}\Big)p^3 + \cdots + \Big(\sum_{i=1}^{m} a_{i,n}\Big)p^n = p^n,$$

then

$$\Big(\sum_{i=1}^{m} a_{i,2}\Big) + \Big(\sum_{i=1}^{m} a_{i,3}\Big)p + \cdots + \Big(\sum_{i=1}^{m} a_{i,n}\Big)p^{n-2} = p^{n-2},$$

so $p | \sum_{i=1}^{m} a_{i,2}$. Note that $a_{i,2} = 0$ or 1, $m \leq p$. It is easy to know that $\sum_{i=1}^{m} a_{i,2} = 0$ or $\sum_{i=1}^{m} a_{i,2} = p$.

Subcase 2.1 $\quad \sum_{i=1}^{m} a_{i,2} = 0$.

If $n = 3$, it is easy to get that $\sum_{i=1}^{m} a_{i,3} = 1$, so there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's.

If $n > 3$, then $(\sum_{i=1}^{m} a_{i,3})p^3 + (\sum_{i=1}^{m} a_{i,4})p^4 + \cdots + (\sum_{i=1}^{m} a_{i,n})p^n = p^n$, so

$$\Big(\sum_{i=1}^{m} a_{i,3}\Big) + \Big(\sum_{i=1}^{m} a_{i,4}\Big)p + \Big(\sum_{i=1}^{m} a_{i,5}\Big)p^2 + \cdots + \Big(\sum_{i=1}^{m} a_{i,n}\Big)p^{n-3} = p^{n-3}.$$

Similarly we know that $\sum_{i=1}^{m} a_{i,3} = 0$ or $\sum_{i=1}^{m} a_{i,3} = p$. We claim that $\sum_{i=1}^{m} a_{i,3} = 0$, for otherwise, we would have $\sum_{i=1}^{m} a_{i,3} = p$, then $m = p$. Then for each $1 \leq i \leq p$, $\dim x_i = 1$ and $\deg x_i =$ higher terms $+p^3 q+$lower terms. Since $\sum_{i=1}^{p} e_i = p - 4$, $\deg a_i \equiv 1 \,(\mathrm{mod}\ q)$ $(i \geq 0)$ and $\deg h_{i,j} \equiv 0 \,(\mathrm{mod}\ q)$ $(i > 0, j \geq 0)$, then there exists a factor $a_{j_1} a_{j_2} \cdots a_{j_{p-4}}$ among x_i's such that for each $1 \leq i \leq p-4$, $j_i \geq 4$ and $\deg a_{j_i} =$ higher terms $+p^3 q + p^2 q + pq + q + 1$. It is obvious that $\sum_{i=1}^{m} a_{i,2} \geq p - 4$ which contradicts $\sum_{i=1}^{m} a_{i,2} = 0$, thus the claim follows. By induction on j, we can get that $\sum_{i=1}^{m} a_{i,j} = 0$ $(3 \leq j \leq n-1)$, so $\sum_{i=1}^{m} a_{i,n} = 1$, that is to say, there is a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's.

All in all, at this time for $n \geq 3$, there is a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's. We denote $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$ (up to sign) or $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$ (up to sign).

If $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$, $h' = x_1 x_2 \cdots x_{m-1} \in E_1^{p-1,t''-p^n q,*}$. By an argument similar to that used in the proof of Theorem 2.2, we can show that $E_1^{s+3,t''-p^n q,*} = 0$, so h' is impossible to exist. Thus h is impossible to be of the form $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$.

If $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$, $h'' = x_1 x_2 x_3 \cdots x_{m-1} \in E_1^{p-2,t''-p^n q,*}$. By Lemma 2.1, we can know that $E_1^{p-2,t''-p^n q,*} = 0$, so $h'' = x_1 x_2 x_3 \cdots x_{m-1}$ is impossible to exist and h is impossible to be of the form $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$.

Subcase 2.2 $\quad$ If $\sum_{i=1}^{m} a_{i,2} = p$, then $m = p$. Since $\dim h = p$, we can easily see that for each i, $\dim x_i = 1$ and $h = x_1 x_2 \cdots x_p \in E(h_{m,i} | m > 0, i \geq 0) \bigotimes P(a_n | n \geq 0)$.

If $n = 3$, we can easily get that $\sum_{i=1}^{p} a_{i,2} = p$, $\sum_{i=1}^{p} a_{i,1} = p - 1$, $\sum_{i=1}^{p} a_{i,0} = p - 3$ and $\sum_{i=1}^{p} e_i = p - 4$.

If $n > 3$, from the equality $(\sum_{i=1}^{p} a_{i,2})p^2 + \cdots + (\sum_{i=1}^{p} a_{i,n})p^n = p^n$, we can have

$$\Big(\sum_{i=1}^{p} a_{i,3} + 1\Big) + \Big(\sum_{i=1}^{p} a_{i,4}\Big)p + \cdots + \Big(\sum_{i=1}^{p} a_{i,n}\Big)p^n = p^n.$$

Then $p|(\sum_{i=1}^{p} a_{i,3}+1)$. Noting that $a_{i,3}=0$ or 1, we have that $\sum_{i=1}^{p} a_{i,3}=p-1$. By induction on j, we can prove that $\sum_{i=1}^{p} a_{i,j}=p-1$ $(3\le j\le n-1)$. So $\sum_{i=1}^{p} a_{i,n}=0$.

When $n=3$, by the facts that $\sum_{i=1}^{p} e_i=p-4$, $\sum_{i=1}^{p} a_{i,0}=p-3$, $\sum_{i=1}^{p} a_{i,1}=p-1$, $\sum_{i=1}^{p} a_{i,2}=p$, we can prove that $h=x_1x_2\cdots x_p$ is impossible to exist by an argument similar to that used in the proof of Theorem 2.2.

When $n>3$, by the facts that $\sum_{i=1}^{p} a_{i,2}=p$, $\sum_{i=1}^{p} a_{i,3}=\cdots=\sum_{i=1}^{p} a_{i,n-1}=p-1$, $\deg h_{k,j}=q(p^{k+j-1}+\cdots+p^j)$ $(k\ge 1, j\ge 0)$ and $\deg a_i=q(p^{i-1}+\cdots+p+1)+1$ $(i>0)$, we can divide the p x_i's into two disjoint classes S_1 and S_2. The two disjoint classes are given by

$$S_1=\{x|\deg x=q(p^{n-1}+p^{n-2}+\cdots+p^2)+\text{lower terms}\},$$
$$S_2=\{x|\deg x=qp^2+\text{lower terms}\}.$$

For a class S, denote the number of elements in S by $N(S)$, then we can get $N(S_1)=p-1$ and $N(S_2)=1$. Similarly, by the facts that $\sum_{i=1}^{p} e_i=p-4$, $\sum_{i=1}^{p} a_{i,0}=p-3$, $\sum_{i=1}^{p} a_{i,1}=p-1$, $\sum_{i=1}^{p} a_{i,2}=p$, $\deg h_{k,j}=q(p^{k+j-1}+\cdots+p^j)$ $(k\ge 1, j\ge 0)$ and $\deg a_i=q(p^{i-1}+\cdots+p+1)+1$ $(i>0)$, we can also divide the p x_i's into four disjoint classes. The four classes are given by

$$\begin{aligned}
&S_3=\{x|\deg x=q(\text{higher terms}+p^2+p+1)+1\}, && N(S_3)=p-4,\\
&S_4=\{x|\deg x=q(\text{higher terms}+p^2+p+1)\}, && N(S_4)=1,\\
&S_5=\{x|\deg x=q(\text{higher terms}+p^2+p)\}, && N(S_5)=2,\\
&S_6=\{x|\deg x=q(\text{higher terms}+p^2)\}, && N(S_6)=1.
\end{aligned}$$

If $S_5\subset S_1$ (i.e., all elements in S_5 are in S_1), then there would be two $h_{n-1,1}$'s such that $\deg h_{n-1,1}=q(p^{n-1}+\cdots+p^3+p^2+p)$. This is impossible since $h_{n-1,1}^2=0$. If $S_5\not\subset S_1$ (i.e., not all elements in S_5 are in S_1), then one of the two elements in S_5 must be in S_2. The element must be $h_{2,1}$ such that $\deg h_{2,1}=q(p^2+p)$. For two classes A and B, define $A\cup B=\{\text{x}|\ x \text{ is in } A \text{ or } x \text{ is in } B\}$. It follows that $S_1\cup S_2=S_3\cup S_4\cup S_5\cup S_6$, so we have $S_6\subset S_1$,$S_4\subset S_1$, $S_3\subset S_1$ and another element in S_5 must be in S_1. We easily get that $S_3=\{\underbrace{a_n,a_n,\cdots,a_n}_{p-4}\}$, $S_4=\{h_{n,0}\}$, $S_5=\{h_{2,1},h_{n-1,1}\}$ and $S_6=\{h_{n-2,2}\}$. Thus $h=\underbrace{a_na_n\cdots a_n}_{p-4}h_{n,0}h_{n-1,1}h_{n-2,2}h_{2,1}$ (up to sign).

From Subcase 2.1 and Subcase 2.2, we see that when $s=p-4$,

$$E_1^{p,t'',*}=\begin{cases} Z_p\{\underbrace{a_na_n\cdots a_n}_{p-4}h_{n,0}h_{n-1,1}h_{n-2,2}h_{2,1}\}, & \text{if } n>3,\\ 0, & \text{if } n=3.\end{cases}$$

When $n=3$, $E_r^{p,t'',*}=0$ $(r\ge 1)$, then $d_r(E_r^{p,t'',*})=0$.

When $n>3$, consider the May filtration of elements $\underbrace{a_2\,a_2\cdots\,a_2}_{p-4}h_{2,0}\ h_{1,1}\ b_{1,0}\ h_{1,n}$ and $\underbrace{a_n\ a_n\cdots\ a_n}_{p-4}h_{n,0}\ h_{n-1,1}\ h_{n-2,2}\ h_{2,1}$. We see that

$$M(\underbrace{a_2a_2\cdots a_2}_{p-4}h_{2,0}h_{1,1}b_{1,0}h_{1,n})=6p-15=M,$$
$$M(\underbrace{a_na_n\cdots a_n}_{p-4}h_{n,0}h_{n-1,1}h_{n-2,2}h_{2,1})=(2n+1)p-2n-10=M+r \quad \text{with } r>2.$$

Now

$$d_1(\underbrace{a_na_n\cdots a_n}_{p-4}h_{n,0}h_{n-1,1}h_{n-2,2}h_{2,1})=\underbrace{a_na_n\cdots a_n}_{p-4}h_{n,0}h_{n-1,1}h_{n-2,2}h_{1,2}h_{1,1}+\cdots\neq 0.$$

Thus $E_2^{p,t'',*}=0$ and no higher May differential hits $\underbrace{a_2a_2\cdots a_2}_{p-4}h_{2,0}h_{1,1}b_{1,0}h_{1,n}$ in the MSS.

This shows that $\tilde{\beta}_{p-2}b_0h_n \neq 0 \in \mathrm{Ext}_A^{p+1,t''}(Z_p, Z_p)$.

From Case 1 and Case 2, the proposition follows.

Proposition 2.4 *Let $p \geq 5$, $n \geq 3$, $0 \leq s < p-3$, $2 \leq r \leq s+5$. Then we have*

$$\mathrm{Ext}_A^{s+5-r,q(p^n+(s+3)p+(s+1))+(s-r+1)}(Z_p, Z_p) = 0.$$

Proof It suffices to prove that in the MSS $E_1^{s+5-r,t''',*} = 0$, where $t''' = q(p^n + (s+3)p + (s+1)) + (s-r+1)$. Suppose that $h = x_1x_2\cdots x_m$ is the generator of $E_1^{s+5-r,t''',*}$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Let deg $x_i = q(a_{i,n}p^n + a_{i,n-1}p^{n-1} + \cdots + a_{i,0}) + e_i$, where $a_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}
\deg h &= \sum_{i=1}^{m} \deg x_i \\
&= q\Bigg(\Bigg(\sum_{i=1}^{m} a_{i,n}\Bigg)p^n + \cdots + \Bigg(\sum_{i=1}^{m} a_{i,2}\Bigg)p^2 + \Bigg(\sum_{i=1}^{m} a_{i,1}\Bigg)p + \Bigg(\sum_{i=1}^{m} a_{i,0}\Bigg)\Bigg) + \Bigg(\sum_{i=1}^{m} e_i\Bigg) \\
&= q(p^n + (s+3)p + (s+1)) + s - r + 1, \\
\dim h &= \sum_{i=1}^{m} \dim x_i = s + 5 - r.
\end{aligned}$$

Noting that dim $x_i = 1$ or 2, we have that $m \leq s+5-r \leq s+3 < p$ from $\sum_{i=1}^m \dim x_i = s+5-r$. We claim that $s-r+1 \geq 0$, otherwise, we would have $p > \sum_{i=1}^m e_i = q + (s-r+1) \geq q-4 \geq p$. That is impossible. The claim follows.

Noting the suppositions that $a_{i,j} = 0$ or 1, $e_i = 0$ or 1 and $m < p$, we have

$$\sum_{i=1}^{m} e_i = s-r+1, \qquad \sum_{i=1}^{m} a_{i,0} = s+1, \qquad \sum_{i=1}^{m} a_{i,1} = s+3,$$
$$\sum_{i=1}^{m} a_{i,2} = 0, \qquad \sum_{i=1}^{m} a_{i,3} = \cdots = \sum_{i=1}^{m} a_{i,n-1} = 0, \qquad \sum_{i=1}^{m} a_{i,n} = 1.$$

It is easy to see that there exists an $h_{1,n}$ or $b_{1,n-1}$ among x_i's. We denote $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h = x_1x_2\cdots x_{m-1}h_{1,n}$ or $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$.

If $h = x_1x_2\cdots x_{m-1}h_{1,n}$, $h' = x_1x_2\cdots x_{m-1} \in E_1^{s+4-r,t'''-p^nq,*} = 0$ by Lemma 2.1. Thus h is impossible to be of the form $h = x_1x_2\cdots x_{m-1}h_{1,n}$.

Similarly, we can show that h is impossible to be of the form $h = x_1x_2\cdots x_{m-1}b_{1,n-1}$ by Lemma 2.1.

From the above discussion we see that $E_1^{s+5-r,t''',*} = 0$, so $\mathrm{Ext}_A^{s+5-r,t'''}(Z_p, Z_p) = 0$. This finishes the proof of Proposition 2.4.

Proposition 2.5 *Let $p \geq 5$, $n \geq 3$, $0 \leq s < s-3$. Then we have*

$$\tilde{\beta}_{s+2}h_1b_{n-1} = 0 \in \mathrm{Ext}_A^{s+5,p^nq+(s+3)pq+(s+1)q+s}(Z_p, Z_p).$$

Proof Since $h_{1,1}^2 = 0 \in E_1^{2,2pq,*}$, then

$$\underbrace{a_2\cdots a_2}_{s} h_{2,0}h_{1,1}h_{1,1}b_{1,n-1} = 0 \in E_1^{s+5,p^nq+(s+3)pq+(s+1)q+s,*},$$

so $\tilde{\beta}_{s+2}h_1b_{n-1} = 0 \in \mathrm{Ext}_A^{s+5,p^nq+(s+3)pq+(s+1)q+s}(Z_p, Z_p)$.

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

Theorem 1.1 *Let $p \geq 5, n \geq 3$. Then*

$$\tilde{\beta}_{s+2}b_0h_n \neq 0 \in \mathrm{Ext}_A^{s+5,p^nq+(s+3)pq+(s+1)q+s}(Z_p, Z_p)$$

is a permanent cycle in the Adams Spectral Sequence and converges to a nontrivial element in $\pi_{p^nq+(s+3)pq+(s+1)q-5}$, *where* $0 \le s < p-3$, $q = 2(p-1)$.

Proof From [3, Theorem A], we get that $i_*(h_1h_n) \in \mathrm{Ext}_A^{2,p^nq+pq}(H^*M, Z_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\xi \in \pi_{p^nq+pq-2}M$. At the same time $j\xi_n \in \pi_{p^nq+pq-3}S$ is a nontrivial element of order p which is represented (up to a nonzero scalar) by $(b_0h_n + h_1b_{n-1}) \in \mathrm{Ext}_A^{3,p^nq+pq}(Z_p, Z_p)$ in the ASS.

Consider the following composition of maps:

$$\overline{\overline{f}} : \Sigma^{p^nq+pq-3}S \xrightarrow{j\xi_n} S \xrightarrow{jj'j\beta^{s+2}i'i} \Sigma^{-2(s+2)(p^2-1)+q+2}S.$$

Since $j\xi_n$ is represented (up to a nonzero scalar) by $(b_0h_n + h_1b_{n-1}) \in \mathrm{Ext}_A^{3,p^nq+pq}(Z_p, Z_p)$, then the above $\overline{\overline{f}}$ is represented (up to a nonzero scalar) by $\overline{\overline{c}} = (jj'\beta^{s+2}i'i)_*(b_0h_n + h_1b_{n-1})$.

From Theorem 2.2 and the knowledge of Yoneda products we know that the composition

$$\mathrm{Ext}_A^{0,0}(Z_p, Z_p) \xrightarrow{(i'i)_*} \mathrm{Ext}_A^{0,0}(H^*M, Z_p) \xrightarrow{(jj')_*(\beta_*)^{s+2}} \mathrm{Ext}_A^{s+2,(s+2)pq+(s+1)q+s}(Z_p, Z_p)$$

is a multiplication (up to a nonzero scalar) by $\tilde{\beta}_{s+2} \in \mathrm{Ext}_A^{s+2,(s+2)pq+(s+1)q+s}(Z_p, Z_p)$. Hence, $\overline{\overline{f}}$ is represented (up to a nonzero scalar) by $\overline{\overline{c}} = \tilde{\beta}_{s+2}(b_0h_n + h_1b_{n-1}) = \tilde{\beta}_{s+2}b_0h_n \ne 0 \in \mathrm{Ext}_A^{s+5,q(p^n+(s+3)p+(s+1))+s)}(Z_p, Z_p) = 0$ in the ASS (cf. Proposition 2.3 and Proposition 2.5).

Moreover, from Proposition 2.4, $\mathrm{Ext}_A^{s+5-r,q(p^n+(s+3)p+(s+1))+(s-r+1)}(Z_p, Z_p) = 0$ for $r \ge 2$, then $\tilde{\beta}_{s+2}b_0h_n$ cannot be hit by the differentials in the ASS, and so the corresponding homotopy element $\overline{\overline{f}} \in \pi_*S$ is nontrivial and of order p. This finishes the proof of Theorem 1.1.

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