LIN JINKUN'S WORK ON THE ADAMS SPECTRAL SEQUENCE UR TOPOLOGY SEMINAR NOVEMBER 10, 2006

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LIN'S BIG THEOREM

Theorem 1 (Lin [Lin03]). For each prime $p \ge 5$ and each integer $n \ge 0$, the Hopf invariant one element

$$h_n \in \operatorname{Ext}_A^{1,p^n q}(H^*(K), \mathbf{Z}/(p))$$

is a permanent cycle and therefore represents a map

$$\omega_n: S^{p^n q - 1} \to K.$$

Here q = 2p - 2 and K is Toda's 4-cell complex V(1).

BACKGROUND

The Adams spectral sequence [Ada58] is a method for computing the graded group of maps $[X, Y]_*$ for suitable spectra X and Y. Its E_2 -term is

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(H^*(Y), H^*(X))$$

where A is the mod p Steenrod algebra, and the cohomology groups have coefficients in $\mathbf{Z}/(p)$. This group is contravariant in X and covariant in Y.

We will abbreviate

$$\operatorname{Ext}(H^*(Y)) := \operatorname{Ext}_A(H^*(Y), \mathbf{Z}/(p)),$$

the E_2 -term for $\pi_*(Y)$

and Ext :=
$$\operatorname{Ext}(\mathbf{Z}/(p)),$$

the E_2 -term for $\pi_*(S^0)$

Ext¹ has basis dual to the set of algebra generators for the mod p Steenrod algebra A, the Bockstein operation Δ and the reduced power operations \mathcal{P}^{p^n} for $n \geq 0$, giving elements

$$a_0 \in \operatorname{Ext}^{1,1}$$
 and $h_n \in \operatorname{Ext}^{1,p^n q}$.

DOUG RAVENEL

 Ext^2 has basis dual to the set of algebra relations for the mod p Steenrod algebra A. In particular for each n > 0, there is an element $b_{n-1} \in \operatorname{Ext}^{2,p^n q}$ corresponding to the relation

$$\left(\mathcal{P}^{p^{n-1}}\right)^{p} = \dots$$

$$a_{0} \in \operatorname{Ext}^{1,1} \quad h_{n} \in \operatorname{Ext}^{1,p^{n}q}$$

$$b_{n-1} \in \operatorname{Ext}^{2,p^{n}q}$$

The following facts about these elements are known.

 a_0 corresponds to p times the identity map ι and h_0 corresponds to the map $\alpha_1 : S^{q-1} \to S^0$.

Liulevicius [Liu62] (using methods introduced by Adams [Ada60] for p = 2) showed that for p odd, h_n for n > 0 is not a permanent cycle, but instead there is a nontrivial differential

$$d_2(h_n) = a_0 b_{n-1}.$$

I showed [Rav78] that for $p \ge 5$ and $n \ge 2$,

$$d_{2p-1}(b_{n-1}) = h_0 b_{n-2}^p \neq 0.$$

Ralph Cohen [Coh81] showed that for $p \ge 3$ and $n \ge 1$, $h_0 b_{n-1}$ is a permanent cycle represented by a map

$$\zeta_{n-1}: S^{N-3} \to S^0$$

where $N = (p^n + 1)q$. This is the geometric input for Lin's theorem.

The mod p Moore spectrum M

M is known to be a ring spectrum for $p \geq 3$. There is a cofiber sequence

$$S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \xrightarrow{p\iota} S^1$$

The element $i_*(a_0)$ vanishes in $\operatorname{Ext}(H^*(M))$, so (2) above is most here, but we do have

(3') There is a differential

$$d_{2p-1}(i_*(h_n)) = a_1 i_*(b_{n-2})^p$$

where $a_1 \in \operatorname{Ext}^{1,q+1}(H^*(M))$ corresponds to the composite

$$S^q \xrightarrow{i} \Sigma^q M \xrightarrow{\alpha} M$$

and α is the Adams self-map.

(4') Cohen [Coh81] showed that for $p \ge 3$ and $n \ge 1$, $i_*(h_0h_n)$ is a permanent cycle.

TODA'S 4-CELL COMPLEX K = V(1)

There is a cofiber sequence

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M$$

and K is known to be a ring spectrum for $p \geq 5$.

Lin's Theorem says that

$$(i'i)_*(h_n) \in \operatorname{Ext}^{1,p^nq}(H^*(K))$$

is a permanent cycle.

Consequences of Lin's Theorem

Assuming Lin's theorem is true, let

$$\omega_n: S^{p^n q - 1} \to K$$

denote a map detected by $(i'i)_*(h_n)$.

*2 There is a well known map (due to Toda)

$$\beta: \Sigma^{(p+1)q} K \to K$$

whose iterates are all nontrivial.*3 Lin shows that for n > 2 and $0 < s < p^{n-2}$, the $\operatorname{composite}$

$$S^{p^nq-1} \xrightarrow{\omega_n} K \xrightarrow{\beta^s} \Sigma^{-s(p+1)q} K \xrightarrow{jj'} S^{q+2-s(p+1)q}$$

is nontrivial and is detected by the element $\gamma_{p^{n-2}/p^{n-2}-s}$ in the Adams-Novikov spectral sequence.

(3') may imply that for n > 2,

$$S^{p^nq-1} \xrightarrow{\omega_n} K \xrightarrow{j'} \Sigma^{q+1}M$$

is detected by

$$i_*(b_{n-2}^p) \in \operatorname{Ext}^{2p,p^nq}(H^*(M))$$

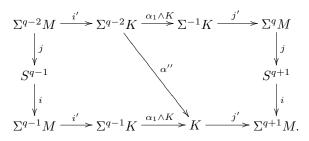
and $j'\omega_n$ may lift to S^{q+1} . For n = 2, the composite

$$S^{p^2q-1} \xrightarrow{\omega_2} K \xrightarrow{jj'} S^{q+2}$$

is $\gamma_1 = \alpha_1 \beta_{p-1} \in \pi_{p^2q-q-3}(S^0)$. For $n = 1, \ j' \omega_1 = 0$ and we have

STEPS IN LIN'S ARGUMENT

Proposition 2 (Toda, [Tod71]). There is a map $\alpha'': \Sigma^{q-2}K \to K$ with



Theorem 3 (Theorem 3.4 of [Lin03]). Let $a'' \in \text{Ext}^{1,q-1}(H^*(K), H^*(K))$ detect α'' . Then

$$(h_0h_n)'' = a''(i'i)_*(h_n) \in \operatorname{Ext}^{2,N-1}(H^*(K),H^*(K))$$

(where $N = (p^n + 1)q$) is a permanent cycle detecting a map

$$\eta_n'': \Sigma^{N-3} K \to K$$

The proof of this is very difficult.

For the next step we need a minimal Adams resolution

$$S^{0} = E_{0} \leftarrow \overline{a_{0}} \Sigma^{-1} E_{1} \leftarrow \overline{a_{1}} \Sigma^{-2} E_{2} \leftarrow \cdots$$

$$\downarrow \overline{b_{0}} \qquad \qquad \downarrow \overline{b_{1}} \qquad \qquad \downarrow \overline{b_{2}}$$

$$H/p = KG_{0} \qquad KG_{1} \qquad KG_{2}$$

and the cofiber sequence

$$S^0 \xrightarrow{i'i} K \xrightarrow{r} Y \xrightarrow{\epsilon} S^1.$$

Corollary 4 (Corollary 3.10 of [Lin03]). The map η''_n lifts to a map $\eta''_{n,2} : \Sigma^{N-3}K \to \Sigma^{-2}E_2 \wedge K$, and there is a map $(\eta''_{n,2})_Y$ making the following diagram commute.

$$\begin{array}{c} \Sigma^{N}K \xrightarrow{r} \Sigma^{N}Y \xrightarrow{(\eta_{n,2}')_{Y}} \Sigma E_{2} \wedge K \\ & \downarrow^{\eta_{n,2}'} & \downarrow^{\overline{b}_{2} \wedge K} \\ \Sigma E_{2} \wedge K \xrightarrow{\overline{b}_{2} \wedge K} & \Sigma K G_{2} \wedge K \end{array}$$

The proof of this is not difficult.

The next step concerns the cofiber sequence

$$\Sigma^{q-2}K \xrightarrow{\alpha''} K \xrightarrow{w} X \xrightarrow{u} \Sigma^{q-1}K$$

Lemma 5 (Lemma 4.2 of [Lin03]). Modulo higher Adams filtration, the composite

$$\Sigma^{N-3}K \xrightarrow{r} \Sigma^{N-3}Y \xrightarrow{(\eta_n'')_Y} K \xrightarrow{w} X$$

is $\lambda' w(\zeta_{n-1} \wedge K)$ for a nonzero scalar λ' , where ζ_{n-1} is Cohen's map.

In other words, there is a map $f_1^{\prime\prime}$ making the following diagram commute

$$\begin{array}{c} \Sigma^{N}K \xrightarrow{r} \Sigma^{N}Y \\ & \downarrow^{(\eta_{n}'')_{Y}} \\ & \downarrow^{(\eta_{n}'')_{Y} \\ & \downarrow^{(\eta_{n}'')_{Y} \\ & \downarrow^{(\eta_{n}'')_{Y} \\ & \downarrow^{(\eta_{n}'')_{Y}} \\ & \downarrow^{(\eta_{n}'')_{Y} \\ & \downarrow^{(\eta_{n}$$

where $\overline{a}_{0,3} = \overline{a}_0 \overline{a}_1 \overline{a}_2 \overline{a}_3$.

The proof of this is difficult, as is the derivation of Theorem 1 from it. The latter involves studying a diagram that uses the cofiber sequences

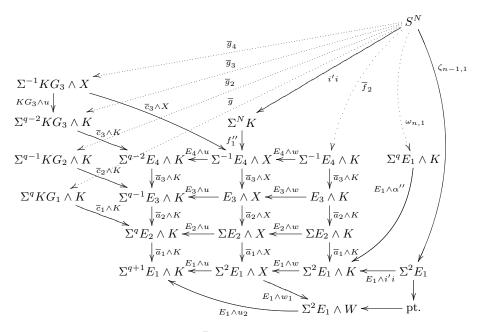
$$\Sigma^{-1}E_{s+1} \xrightarrow{\overline{a}_s} E_s \xrightarrow{\overline{b}_s} KG_s \xrightarrow{\overline{c}_s} E_{s+1},$$

$$S^0 \xrightarrow{wi'i} X \xrightarrow{w_1} W \xrightarrow{u_1} S^1$$

$$\Sigma^{q-2}K \xrightarrow{r\alpha''} V \xrightarrow{w_2} W \xrightarrow{u_2} \Sigma^{q-1}K$$

and

$$\Sigma^{q-2}K \xrightarrow{r\alpha''} Y \xrightarrow{w_2} W \xrightarrow{u_2} \Sigma^{q-1}K.$$



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