# Lecture 6: The detection theorem

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
Tokyo City University

November 6, 2009


Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester

# $\theta_j$ in the Adams-Novikov spectral sequence

Browder's Theorem says that $\theta_j$ is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$

# $\theta_j$ in the Adams-Novikov spectral sequence

Browder's Theorem says that $\theta_j$ is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$

This element is known to be the only one in its bidegree.



Mike Hill
Mike Hopkins
Doug Ravenel

# $\theta_j$ in the Adams-Novikov spectral sequence

Browder's Theorem says that $\theta_j$ is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence.

# $\theta_j$ in the Adams-Novikov spectral sequence

Browder's Theorem says that $\theta_j$ is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \mathrm{Ext}_{MU_*(MU)}^{2,6i-2j}(MU_*, MU_*)$$

for certain values of of $i$ and $j$. When $j = 1$, it is customary to omit it from the notation.

# $\theta_j$ in the Adams-Novikov spectral sequence

Browder's Theorem says that $\theta_j$ is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \mathrm{Ext}_{MU_*(MU)}^{2,6i-2j}(MU_*, MU_*)$$

for certain values of of $i$ and $j$. When $j = 1$, it is customary to omit it from the notation. The definition of these elements can be found in Chapter 5 of the third author's book Complex Cobordism and Stable Homotopy Groups of Spheres.

# $\theta_j$ in the Adams-Novikov spectral sequence (continued)

Here are the first few of these in the relevant bidegrees.

$$\begin{array}{ll} \theta_5 : & \beta_{8/8} \text{ and } \beta_{6/2} \\ \theta_6 : & \beta_{16/16},\ \beta_{12/4} \text{ and } \beta_{11} \\ \theta_7 : & \beta_{32/32},\ \beta_{24/8} \text{ and } \beta_{22/2} \\ \theta_8 : & \beta_{64/64},\ \beta_{48/16},\ \beta_{44/4} \text{ and } \beta_{43} \end{array}$$

and so on.



**Mike Hill**
**Mike Hopkins**
**Doug Ravenel**

# $\theta_j$ in the Adams-Novikov spectral sequence (continued)

Here are the first few of these in the relevant bidegrees.

$$\begin{array}{ll}
\theta_5: & \beta_{8/8} \text{ and } \beta_{6/2} \\
\theta_6: & \beta_{16/16},\ \beta_{12/4} \text{ and } \beta_{11} \\
\theta_7: & \beta_{32/32},\ \beta_{24/8} \text{ and } \beta_{22/2} \\
\theta_8: & \beta_{64/64},\ \beta_{48/16},\ \beta_{44/4} \text{ and } \beta_{43}
\end{array}$$

and so on. In the bidegree of $\theta_j$, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely $h_j^2$) in the Adams spectral sequence.

# $\theta_j$ in the Adams-Novikov spectral sequence (continued)

Here are the first few of these in the relevant bidegrees.

$$\begin{array}{ll} \theta_5: & \beta_{8/8} \text{ and } \beta_{6/2} \\ \theta_6: & \beta_{16/16},\ \beta_{12/4} \text{ and } \beta_{11} \\ \theta_7: & \beta_{32/32},\ \beta_{24/8} \text{ and } \beta_{22/2} \\ \theta_8: & \beta_{64/64},\ \beta_{48/16},\ \beta_{44/4} \text{ and } \beta_{43} \end{array}$$

and so on. In the bidegree of $\theta_j$, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely $h_j^2$) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1\alpha_{2^j-1}$.

# $\theta_j$ in the Adams-Novikov spectral sequence (continued)

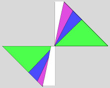
Here are the first few of these in the relevant bidegrees.

$$\begin{array}{ll} \theta_5 : & \beta_{8/8} \text{ and } \beta_{6/2} \\ \theta_6 : & \beta_{16/16},\ \beta_{12/4} \text{ and } \beta_{11} \\ \theta_7 : & \beta_{32/32},\ \beta_{24/8} \text{ and } \beta_{22/2} \\ \theta_8 : & \beta_{64/64},\ \beta_{48/16},\ \beta_{44/4} \text{ and } \beta_{43} \end{array}$$

and so on. In the bidegree of $\theta_j$, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely $h_j^2$) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1\alpha_{2^j-1}$.

We need to show that any element mapping to $h_j^2$ in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for $\Omega$.





# $\theta_j$ in the Adams-Novikov spectral sequence (continued)

## Detection Theorem

*Let $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}^{2,2^{j+1}}_{A}(\mathbf{Z}/2, \mathbf{Z}/2)$ is $h_j^2$ with $j \geq 6$.*

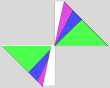


Mike Hill
Mike Hopkins
Doug Ravenel

# $\theta_j$ in the Adams-Novikov spectral sequence (continued)



Mike Hill
Mike Hopkins
Doug Ravenel





**Detection Theorem**

*Let $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}^{2,2^{j+1}}_A(\mathbf{Z}/2, \mathbf{Z}/2)$ is $h_j^2$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.)*

# $\theta_j$ in the Adams-Novikov spectral sequence (continued)

**Detection Theorem**

*Let $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}^{2,2^{j+1}}_{A}(\mathbf{Z}/2, \mathbf{Z}/2)$ is $h_j^2$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.) Then the image of $x$ in $H^{2,2^{j+1}}(C_8; \pi_*(\Omega))$ is nonzero.*

# $\theta_j$ in the Adams-Novikov spectral sequence (continued)

**Detection Theorem**

*Let* $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ *be any element whose image in* $\mathrm{Ext}^{2,2^{j+1}}_{A}(\mathbf{Z}/2, \mathbf{Z}/2)$ *is* $h_j^2$ *with* $j \geq 6$*. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in* $H^{2,2^{j+1}}(C_8; \pi_*(\Omega))$ *is nonzero.*

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, the theory of formal $A$-modules, where $A$ is the ring of integers in a suitable field.

# Formal $A$-modules

Recall the a formal group law over a ring $R$ is a power series

$$F(x, y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x, y]]$$

with certain properties.



**Mike Hill**
**Mike Hopkins**
**Doug Ravenel**

# Formal $A$-modules

Recall the a formal group law over a ring $R$ is a power series

$$F(x, y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x, y]]$$

with certain properties.

For positive integers $m$ one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

# Formal $A$-modules

Recall the a formal group law over a ring $R$ is a power series

$$F(x,y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x,y]]$$

with certain properties.

For positive integers $m$ one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

# Formal $A$-modules

Recall the a formal group law over a ring $R$ is a power series

$$F(x,y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x,y]]$$

with certain properties.

For positive integers $m$ one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers $m$, and we get a homomorphism $\tau$ from $\mathbf{Z}$ to $End(F)$, the endomorphism ring of $F$.

# Formal $A$-modules (continued)

If the ground ring $R$ is an algebra over the $p$-local integers $\mathbf{Z}_{(p)}$ or the $p$-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for $m$ in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.



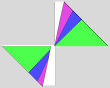
Mike Hill
Mike Hopkins
Doug Ravenel

# Formal $A$-modules (continued)

If the ground ring $R$ is an algebra over the $p$-local integers $\mathbf{Z}_{(p)}$ or the $p$-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for $m$ in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.

Now suppose $R$ is an algebra over a larger ring $A$, such as the ring of integers in a number field or a finite extension of the $p$-adic numbers.





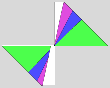
# Formal $A$-modules (continued)

If the ground ring $R$ is an algebra over the $p$-local integers $\mathbf{Z}_{(p)}$ or the $p$-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for $m$ in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.

Now suppose $R$ is an algebra over a larger ring $A$, such as the ring of integers in a number field or a finite extension of the $p$-adic numbers. We say that the formal group law $F$ is a formal $A$-module if the homomorphism $\tau$ extends to $A$ in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$





# Formal $A$-modules (continued)

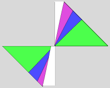
If the ground ring $R$ is an algebra over the $p$-local integers $\mathbf{Z}_{(p)}$ or the $p$-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for $m$ in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.

Now suppose $R$ is an algebra over a larger ring $A$, such as the ring of integers in a number field or a finite extension of the $p$-adic numbers. We say that the formal group law $F$ is a formal $A$-module if the homomorphism $\tau$ extends to $A$ in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$

The theory of formal $A$-modules is well developed. Lubin-Tate used them to do local class field theory.





# Formal $A$-modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where $\zeta_8$ is a primitive 8th root of unity.

# Formal $A$-modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where $\zeta_8$ is a primitive 8th root of unity. The maximal ideal of $A$ is generated by $\pi = \zeta_8 - 1$, and $\pi^4$ is a unit multiple of 2.

# Formal $A$-modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where $\zeta_8$ is a primitive 8th root of unity. The maximal ideal of $A$ is generated by $\pi = \zeta_8 - 1$, and $\pi^4$ is a unit multiple of 2. There is a formal $A$-module $F$ over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y)$$

where

$$\log_F(x) = \sum_{n \geq 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}.$$



Mike Hill
Mike Hopkins
Doug Ravenel

# Formal $A$-modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where $\zeta_8$ is a primitive 8th root of unity. The maximal ideal of $A$ is generated by $\pi = \zeta_8 - 1$, and $\pi^4$ is a unit multiple of 2. There is a formal $A$-module $F$ over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y)$$

where

$$\log_F(x) = \sum_{n \geq 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}.$$

The classifying map $\lambda : MU_* \to R_*$ for $F$ factors through $BP_*$, where the logarithm is

$$\log(x) = \sum_{n \geq 0} \ell_n x^{2^n}.$$

# Formal $A$-modules (continued)

Recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \dots]$ with $|v_n| = 2(2^n - 1)$.

# Formal $A$-modules (continued)

Recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \ldots]$ with $|v_n| = 2(2^n - 1)$. The $v_n$ and the $\ell_n$ are related by Hazewinkel's formula,

$$\begin{aligned}
\ell_1 &= \frac{v_1}{2} \\
\ell_2 &= \frac{v_2}{2} + \frac{v_1^3}{4} \\
\ell_3 &= \frac{v_3}{2} + \frac{v_1 v_2^2 + v_2 v_1^4}{4} + \frac{v_1^7}{8} \\
\ell_4 &= \frac{v_4}{2} + \frac{v_1 v_3^2 + v_2^5 + v_3 v_1^8}{4} + \frac{v_1^3 v_2^4 + v_1^9 v_2^2 + v_2 v_1^{12}}{8} + \frac{v_1^{15}}{16} \\
&\vdots
\end{aligned}$$

# The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU_{\mathbf{R}}^{(4)}$?

# The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1} MU_{\mathbf{R}}^{(4)}$?
Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$.

# The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU_{\mathbf{R}}^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$.

# The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU_{\mathbf{R}}^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

# The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU_{\mathbf{R}}^{(4)}$?

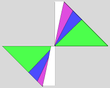
Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

**Lemma**

*The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $F$ factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*



Mike Hill
Mike Hopkins
Doug Ravenel



# The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU_{\mathbf{R}}^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

**Lemma**

*The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $F$ factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \to R_*$ is equivariant, where $C_8$ acts on $\pi_*(MU_{\mathbf{R}}^{(4)})$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.*

# The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1} MU_{\mathbf{R}}^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

**Lemma**

*The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $F$ factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \to R_*$ is equivariant, where $C_8$ acts on $\pi_*(MU_{\mathbf{R}}^{(4)})$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.*
- *The element $D \in \pi_*(MU_{\mathbf{R}}^{(4)})$ that we invert to get $\Omega$ goes to a unit in $R_*$.*

## The relation between $MU^{(4)}$ and formal $A$-modules

What does this have to do with our spectrum $\Omega = D^{-1}MU_{\mathbf{R}}^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

**Lemma**

*The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $F$ factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \to R_*$ is equivariant, where $C_8$ acts on $\pi_*(MU_{\mathbf{R}}^{(4)})$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.*
- *The element $D \in \pi_*(MU_{\mathbf{R}}^{(4)})$ that we invert to get $\Omega$ goes to a unit in $R_*$.*

We will prove this later.

# The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8;\pi_*(D^{-1}MU_{\mathbf{R}}^{(4)})) = H^*(C_8;\pi_*(\Omega)) \to H^*(C_8;R_*).$$

# The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8;\pi_*(D^{-1}MU_{\mathbf{R}}^{(4)})) = H^*(C_8;\pi_*(\Omega)) \to H^*(C_8;R_*).$$

The source here is the $E_2$-term of the homotopy fixed point spectral sequence for $\Omega$, and the target is easy to calculate.

# The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8; \pi_*(D^{-1}MU_{\mathbf{R}}^{(4)})) = H^*(C_8; \pi_*(\Omega)) \to H^*(C_8; R_*).$$

The source here is the $E_2$-term of the homotopy fixed point spectral sequence for $\Omega$, and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

# The proof of the Detection Theorem

It follows that we have a map

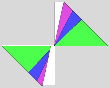
$$H^*(C_8; \pi_*(D^{-1}MU_{\mathbf{R}}^{(4)})) = H^*(C_8; \pi_*(\Omega)) \to H^*(C_8; R_*).$$

The source here is the $E_2$-term of the homotopy fixed point spectral sequence for $\Omega$, and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

**Detection Theorem**

*Let $x \in \mathrm{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2)$ is $h_j^2$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.) Then the image of $x$ in $H^{2,2^{j+1}}(C_8; \pi_*(\Omega))$ is nonzero.*





# The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8; \pi_*(D^{-1}MU_{\mathbf{R}}^{(4)})) = H^*(C_8; \pi_*(\Omega)) \to H^*(C_8; R_*).$$

The source here is the $E_2$-term of the homotopy fixed point spectral sequence for $\Omega$, and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

**Detection Theorem**

*Let $x \in \mathrm{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2)$ is $h_j^2$ with $j \geq 6$. (Here $A$ denotes the mod 2 Steenrod algebra.) Then the image of $x$ in $H^{2,2^{j+1}}(C_8; \pi_*(\Omega))$ is nonzero.*

We will prove this by showing that the image of $x$ in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero.

# The proof of the Detection Theorem (continued)

We will calculate with $BP$-theory.



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Detection Theorem (continued)

We will calculate with $BP$-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \ldots] \quad \text{where } |t_n| = 2(2^n - 1).$$

# The proof of the Detection Theorem (continued)

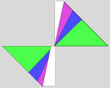
We will calculate with $BP$-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\mathrm{Ext}^{s,t}_{BP_*)(BP)}(BP_*, BP_*)$ by $\mathrm{Ext}^{s,t}$.



Mike Hill
Mike Hopkins
Doug Ravenel



# The proof of the Detection Theorem (continued)

We will calculate with $BP$-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \ldots] \quad \text{where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\mathrm{Ext}^{s,t}_{BP_*)(BP)}(BP_*, BP_*)$ by $\mathrm{Ext}^{s,t}$.

There is a map from this Hopf algebroid to one associated with $H^*(C_8; R_*)$ in which $t_n$ maps to an $R_*$-valued function on $C_8$ (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n\geq 0}^{F} \langle t_n,\, \zeta\rangle x^{2^n}.$$

## The proof of the Detection Theorem (continued)

We will calculate with $BP$-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_n| = 2(2^n - 1).$$

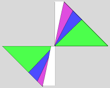
We will abbreviate $\mathrm{Ext}^{s,t}_{BP_*)(BP)}(BP_*, BP_*)$ by $\mathrm{Ext}^{s,t}$.

There is a map from this Hopf algebroid to one associated with $H^*(C_8; R_*)$ in which $t_n$ maps to an $R_*$-valued function on $C_8$ (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n\geq 0}^{F} \langle t_n, \zeta\rangle x^{2^n}.$$

An easy calculation shows that the function $t_1$ sends a primitive root in $C_8$ to a unit in $R_*$.





# The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2} \sum_{0<i<2^j} \binom{2^j}{i} \left[ t_1^i \middle| t_1^{2^j-i} \right] \in \mathrm{Ext}^{2,2^{j+1}}$$

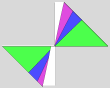
# The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2}\sum_{0<i<2^j}\binom{2^j}{i}\left[t_1^i|t_1^{2^j-i}\right] \in \mathrm{Ext}^{2,2^{j+1}}$$

It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2.





# The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2} \sum_{0<i<2^j} \binom{2^j}{i} \left[ t_1^i | t_1^{2^j - i} \right] \in \mathrm{Ext}^{2,2^{j+1}}$$

It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8; R_*)$ is nontrivial for $j \geq 2$.

# The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2} \sum_{0<i<2^j} \binom{2^j}{i} \left[ t_1^i | t_1^{2^j-i} \right] \in \mathrm{Ext}^{2,2^{j+1}}$$

It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8; R_*)$ is nontrivial for $j \geq 2$.

$H^*(C_8; R_*)$ is the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma-1} R_*[C_8] \xrightarrow{\text{Trace}} R_*[C_8] \xrightarrow{\gamma-1} \cdots$$

where Trace is multiplication by $1 + \gamma + \cdots + \gamma^7$.

# The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2.

# The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$H^1(C_8; R_{2m}) \quad = \quad \ker(1 + \zeta_8^m + \cdots + \zeta_8^{7m})/\operatorname{im}(\zeta_8^m - 1)$$

# The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$\begin{aligned} H^1(C_8; R_{2m}) &= \ker(1 + \zeta_8^m + \cdots + \zeta_8^{7m})/\mathrm{im}\,(\zeta_8^m - 1) \\ &= \begin{cases} w^m A/(\pi) & \text{for } m \text{ odd} \\ w^m A/(\pi^2) & \text{for } m \equiv 2 \bmod 4 \\ w^m A/(2) & \text{for } m \equiv 4 \bmod 8 \\ 0 & \text{for } m \equiv 0 \bmod 8 \end{cases} \end{aligned}$$



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$
\begin{aligned}
H^1(C_8; R_{2m}) &= \ker(1+\zeta_8^m+\cdots+\zeta_8^{7m})/\operatorname{im}(\zeta_8^m-1) \\
&= \begin{cases} w^m A/(\pi) & \text{for } m \text{ odd} \\ w^m A/(\pi^2) & \text{for } m \equiv 2 \bmod 4 \\ w^m A/(2) & \text{for } m \equiv 4 \bmod 8 \\ 0 & \text{for } m \equiv 0 \bmod 8 \end{cases} \\
H^2(C_8; R_{2m}) &= \ker(\zeta_8^m-1)/\operatorname{im}(1+\zeta_8^m+\cdots+\zeta_8^{7m})
\end{aligned}
$$

# The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$
\begin{aligned}
H^1(C_8; R_{2m}) &= \ker\,(1+\zeta_8^m+\cdots+\zeta_8^{7m})/\mathrm{im}\,(\zeta_8^m-1)\\
&= \begin{cases} w^mA/(\pi) & \text{for } m \text{ odd}\\ w^mA/(\pi^2) & \text{for } m\equiv 2 \bmod 4\\ w^mA/(2) & \text{for } m\equiv 4 \bmod 8\\ 0 & \text{for } m\equiv 0 \bmod 8\end{cases}\\
H^2(C_8; R_{2m}) &= \ker\,(\zeta_8^m-1)/\mathrm{im}\,(1+\zeta_8^m+\cdots+\zeta_8^{7m})\\
&= \begin{cases} w^mA/(8) & \text{for } m\equiv 0 \bmod 8\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

## The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in $s$ with period 2. We have

$$
\begin{aligned}
H^1(C_8; R_{2m}) &= \ker\,(1+\zeta_8^m+\cdots+\zeta_8^{7m})/\mathrm{im}\,(\zeta_8^m-1)\\
&= \begin{cases} w^mA/(\pi) & \text{for } m \text{ odd}\\ w^mA/(\pi^2) & \text{for } m\equiv 2 \bmod 4\\ w^mA/(2) & \text{for } m\equiv 4 \bmod 8\\ 0 & \text{for } m\equiv 0 \bmod 8\end{cases}\\
H^2(C_8; R_{2m}) &= \ker\,(\zeta_8^m-1)/\mathrm{im}\,(1+\zeta_8^m+\cdots+\zeta_8^{7m})\\
&= \begin{cases} w^mA/(8) & \text{for } m\equiv 0 \bmod 8\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2^j}$, which is the element of order 2 in $H^2(C_8; R_{2^{j+1}})$.

# The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$.

# The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \leq k < j/2\right\}$$

where $c(j,k) = 2^{j-1-2k}(1+2^{2k+1})/3$.

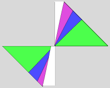
# The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \leq k < j/2\}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.





# The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \leq k < j/2\right\}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.

We will see in the proof of the Lemma below that $v_1$ and $v_2$ map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively.

# The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \leq k < j/2\right\}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.

We will see in the proof of the Lemma below that $v_1$ and $v_2$ map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on $BP_*$ compatible with the one on $A$ in which $||2|| = 1$, $||\pi|| = 1/4$, $||v_1|| = 3/4$ and $||v_2|| = 1/2$.

# The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other $\beta$s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \leq k < j/2\right\}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.

We will see in the proof of the Lemma below that $v_1$ and $v_2$ map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on $BP_*$ compatible with the one on $A$ in which $||2|| = 1$, $||\pi|| = 1/4$, $||v_1|| = 3/4$ and $||v_2|| = 1/2$. We extend the valuation on $A$ to $R_*$ by setting $||w|| = 0$.

# The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$||\beta_{c(j,k)/2^{j-1-2k}}||$$



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\left\| \beta_{c(j,k)/2^{j-1-2k}} \right\| = \left\| \frac{v_2^{c(j,k)}}{2 v_1^{2^{j-1-2k}}} \right\|$$

# The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\begin{aligned} \|\beta_{c(j,k)/2^{j-1-2k}}\| &= \left\| \frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}} \right\| \\ &= \frac{c(j,k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \end{aligned}$$

# The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\begin{aligned}
||\beta_{c(j,k)/2^{j-1-2k}}|| &= \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right| \\
&= \frac{c(j,k)}{2} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\
&= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\
&= (2^{j-1} - 7\cdot 2^{j-3-2k})/3 - 1 \\
&\geq 5.
\end{aligned}$$

# The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\begin{aligned} ||\beta_{c(j,k)/2^{j-1-2k}}|| &= \left|\left| \frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}} \right|\right| \\ &= \frac{c(j,k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \\ &= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \\ &= (2^{j-1} - 7 \cdot 2^{j-3-2k})/3 - 1 \\ &\geq 5. \end{aligned}$$

This means $\beta_{c(j,k))/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero.

# The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1\alpha_{2^j-1}$.

# The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1\alpha_{2^j-1}$. We have

$$\begin{aligned}
||\alpha_{2^j-1}|| &= \left|\left|\frac{v_1^{2^j-1}}{2}\right|\right| \\
&= \frac{3(2^j-1)}{4} - 1 \\
&\geq \frac{21}{4} - 1 \geq 4 \quad \text{for } j \geq 3.
\end{aligned}$$

# The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1\alpha_{2^j-1}$. We have

$$\begin{aligned} ||\alpha_{2^j-1}|| &= \left|\left|\frac{v_1^{2^j-1}}{2}\right|\right| \\ &= \frac{3(2^j-1)}{4} - 1 \\ &\geq \frac{21}{4} - 1 \geq 4 \quad \text{for } j \geq 3. \end{aligned}$$

This completes the proof of the Detection Theorem modulo the Lemma.

# The proof of the Lemma

Here it is again.



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Lemma

Here it is again.

**Lemma**

*The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $F$ factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*

# The proof of the Lemma

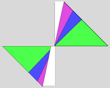
Here it is again.

**Lemma**

*The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for $F$ factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \to R_*$ is equivariant, where $C_8$ acts on $\pi_*(MU_{\mathbf{R}}^{(4)})$ as before, it acts trivially on $A$ and $\gamma w = \zeta_8 w$ for a generator $\gamma$ of $C_8$.*





# The proof of the Lemma

Here it is again.

**Lemma**

*The classifying homomorphism* $\lambda : \pi_*(MU) \to R_*$ *for* $F$ *factors through* $\pi_*(MU_{\mathbf{R}}^{(4)})$ *in such a way that*

- *the homomorphism* $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \to R_*$ *is equivariant, where* $C_8$ *acts on* $\pi_*(MU_{\mathbf{R}}^{(4)})$ *as before, it acts trivially on* $A$ *and* $\gamma w = \zeta_8 w$ *for a generator* $\gamma$ *of* $C_8$*.*
- *The element* $D \in \pi_*(MU_{\mathbf{R}}^{(4)})$ *that we invert to get* $\Omega$ *goes to a unit in* $R_*$*.*

# The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.

$$\begin{tikzcd}
 & MU_*(MU) \arrow[d, equal] & \\
\pi_*(MU) \arrow[ur, "\eta_L"] \arrow[dr, "\lambda_1"'] & \pi_*(MU_{\mathbf{R}}^{(2)}) \arrow[d, dotted, "\lambda^{(2)}"'] & \pi_*(MU) \arrow[ul, "\eta_R"'] \arrow[dl, "\lambda_2"] \\
 & K &
\end{tikzcd}$$



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.

$$\begin{tikzcd}
 & MU_*(MU) \arrow[d, equal] & \\
\pi_*(MU) \arrow[ur, "\eta_L"] \arrow[dr, "\lambda_1"'] & \pi_*(MU_{\mathbf{R}}^{(2)}) \arrow[d, dotted, "\lambda^{(2)}"'] & \pi_*(MU) \arrow[ul, "\eta_R"'] \arrow[dl, "\lambda_2"] \\
 & K &
\end{tikzcd}$$

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$.

# The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.

$$\begin{tikzcd}
 & MU_*(MU) \arrow[d, equal] & \\
\pi_*(MU) \arrow[ur, "\eta_L"] \arrow[dr, "\lambda_1"'] & \pi_*(MU^{(2)}_{\mathbf{R}}) \arrow[d, dotted, "\lambda^{(2)}"'] & \pi_*(MU) \arrow[ul, "\eta_R"'] \arrow[dl, "\lambda_2"] \\
 & K &
\end{tikzcd}$$

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws.

# The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring $K$.

$$
\begin{tikzcd}
 & MU_*(MU) \arrow[d, equal] & \\
\pi_*(MU) \arrow[ur, "\eta_L"] \arrow[dr, "\lambda_1"'] & \pi_*(MU_{\mathbf{R}}^{(2)}) \arrow[d, dotted, "\lambda^{(2)}"'] & \pi_*(MU) \arrow[ul, "\eta_R"'] \arrow[dl, "\lambda_2"] \\
 & K &
\end{tikzcd}
$$

The maps $\lambda_1$ and $\lambda_2$ classify two formal group laws $F_1$ and $F_2$ over $K$. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a compatible strict isomorphism between $F_1$ and $F_2$.

# The proof of the Lemma (continued)

Similarly consider the diagram

$$\begin{tikzcd}
 & & \pi_*(MU_{\mathbf{R}}^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_1"'] & \pi_*(MU) \arrow[ur] \arrow[dr, "\lambda_2"] & & \pi_*(MU) \arrow[ul] \arrow[dl, "\lambda_3"'] & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}$$

# The proof of the Lemma (continued)

Similarly consider the diagram

$$\begin{tikzcd}
 & & \pi_*(MU_{\mathbf{R}}^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_1"'] & \pi_*(MU) \arrow[ur] \arrow[dr, "\lambda_2"] & & \pi_*(MU) \arrow[ul] \arrow[dl, "\lambda_3"'] & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}$$

The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the formal group laws $F_j$ classified by the $\lambda_j$.

# The proof of the Lemma (continued)

$$
\begin{tikzcd}
 & & & \pi_*(MU_{\mathbf{R}}^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & & \\
\pi_*(MU) \arrow[urrr] \arrow[drrr, "\lambda_1"'] & \pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_2"] & & & & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_3"'] & \pi_*(MU) \arrow[ulll] \arrow[dlll, "\lambda_4"] \\
 & & & K & & &
\end{tikzcd}
$$

Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $MU_{\mathbf{R}}^{(4)}$.

# The proof of the Lemma (continued)

$$\begin{tikzcd}
& & \pi_*(MU_{\mathbf{R}}^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_1"'] & \pi_*(MU) \arrow[ur] \arrow[dr, "\lambda_2"] & & \pi_*(MU) \arrow[ul] \arrow[dl, "\lambda_3"'] & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_4"] \\
& & K & &
\end{tikzcd}$$

Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $MU_{\mathbf{R}}^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending $x$ to its formal inverse on each of the $F_j$.

# The proof of the Lemma (continued)

$$\begin{tikzcd}
 & & \pi_*(MU_{\mathbf{R}}^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_1"'] & \pi_*(MU) \arrow[ur] \arrow[dr, "\lambda_2"] & & \pi_*(MU) \arrow[ul] \arrow[dl, "\lambda_3"'] & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}$$

Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $MU_{\mathbf{R}}^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending $x$ to its formal inverse on each of the $F_j$.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbf{Z}[\zeta_8]$-module structure on each of the $F_j$, which are all isomorphic.

# The proof of the Lemma (continued)

$$
\begin{tikzcd}
 & & \pi_*(MU_{\mathbf{R}}^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_1"'] & \pi_*(MU) \arrow[ur] \arrow[dr, "\lambda_2"] & & \pi_*(MU) \arrow[ul] \arrow[dl, "\lambda_3"'] & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}
$$

Now suppose that $K$ has a $C_8$-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined $C_8$-action on $MU_{\mathbf{R}}^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending $x$ to its formal inverse on each of the $F_j$.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbf{Z}[\zeta_8]$-module structure on each of the $F_j$, which are all isomorphic. This proves the first part of the Lemma.

# The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

# The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

Since our formal $A$-module is 2-typical we can do the calculations using $BP$ in place of $MU$.

# The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

Since our formal $A$-module is 2-typical we can do the calculations using $BP$ in place of $MU$. Hence we can replace $x_{2^k-1}$ by $v_k$ and $r_{2^k-1}$ by $t_k$.

# The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

Since our formal $A$-module is 2-typical we can do the calculations using $BP$ in place of $MU$. Hence we can replace $x_{2^k-1}$ by $v_k$ and $r_{2^k-1}$ by $t_k$. We have $\overline{\Delta}_k^{(2)} = v_k$.

# The proof of the Lemma (continued)

Using Hazewinkel's formula we find that



**Mike Hill**
**Mike Hopkins**
**Doug Ravenel**

# The proof of the Lemma (continued)

Using Hazewinkel's formula we find that

$$v_1 \quad \mapsto \quad (-\pi^3 - 4\pi^2 - 6\pi - 4)w \;=\; \pi^3 \cdot \text{unit} \cdot w$$



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Lemma (continued)

Using Hazewinkel's formula we find that

$$
\begin{aligned}
v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w \;=\; \pi^3 \cdot \mathrm{unit} \cdot w \\
v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \;=\; \pi^2 \cdot \mathrm{unit} \cdot w^3
\end{aligned}
$$

# The proof of the Lemma (continued)

Using Hazewinkel's formula we find that

$$
\begin{aligned}
v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w \;=\; \pi^3 \cdot \text{unit} \cdot w \\
v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \;=\; \pi^2 \cdot \text{unit} \cdot w^3 \\
v_3 &\mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \\
&= \pi \cdot \text{unit} \cdot w^7
\end{aligned}
$$

# The proof of the Lemma (continued)

Using Hazewinkel's formula we find that

$$
\begin{aligned}
v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w \;=\; \pi^3 \cdot \mathrm{unit} \cdot w \\
v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \;=\; \pi^2 \cdot \mathrm{unit} \cdot w^3 \\
v_3 &\mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \\
&= \pi \cdot \mathrm{unit} \cdot w^7 \\
v_4 &\mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15} \\
&= \mathrm{unit} \cdot w^7.
\end{aligned}
$$

# The proof of the Lemma (continued)

Using Hazewinkel's formula we find that

$$\begin{aligned}
v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w \;=\; \pi^3 \cdot \mathrm{unit} \cdot w \\
v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \;=\; \pi^2 \cdot \mathrm{unit} \cdot w^3 \\
v_3 &\mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \\
&= \pi \cdot \mathrm{unit} \cdot w^7 \\
v_4 &\mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15} \\
&= \mathrm{unit} \cdot w^7.
\end{aligned}$$

where each unit is in $A$.

# The proof of the Lemma (continued)

Using Hazewinkel's formula we find that

$$
\begin{aligned}
v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w \;=\; \pi^3 \cdot \text{unit} \cdot w \\
v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \;=\; \pi^2 \cdot \text{unit} \cdot w^3 \\
v_3 &\mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \\
&= \pi \cdot \text{unit} \cdot w^7 \\
v_4 &\mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15} \\
&= \text{unit} \cdot w^7.
\end{aligned}
$$

where each unit is in $A$. It follows that $v_4$ (but not $v_n$ for $n < 4$) and therefore $N_2^8(\overline{\Delta}_4^{(2)})$ maps to a unit.

# The proof of the Lemma (continued)

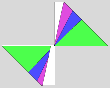
We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \to BP_*^{(4)} \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$





# The proof of the Lemma (continued)

We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \to BP_*^{(4)} \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

Using the right unit formula we find that

# The proof of the Lemma (continued)

We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \to BP_*^{(4)} \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

Using the right unit formula we find that

$$\begin{aligned} t_1 &\mapsto (\pi+2)w = \pi \cdot \mathrm{unit} \cdot w \\ t_2 &\mapsto (\pi^3+5\pi^2+9\pi+5)w^3 = \mathrm{unit} \cdot w^3. \end{aligned}$$

# The proof of the Lemma (continued)

We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \to BP_*^{(4)} \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

Using the right unit formula we find that

$$\begin{aligned} t_1 &\mapsto (\pi+2)w = \pi \cdot \text{unit} \cdot w \\ t_2 &\mapsto (\pi^3+5\pi^2+9\pi+5)w^3 = \text{unit} \cdot w^3. \end{aligned}$$

This means $t_2$ (but not $t_1$) and therefore $N_4^8(\overline{\Delta}_2^{(4)})$ maps to a unit.

# The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$,

# The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$.



Mike Hill
Mike Hopkins
Doug Ravenel

# The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

$$\begin{aligned} \ell_n(1) &\mapsto \frac{w^{2^n-1}}{\pi^n} \\ \ell_n(2) &\mapsto \frac{(\zeta_8 w)^{2^n-1}}{\pi^n}. \end{aligned}$$

# The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

$$\begin{aligned} \ell_n(1) &\mapsto \frac{w^{2^n-1}}{\pi^n} \\ \ell_n(2) &\mapsto \frac{(\zeta_8 w)^{2^n-1}}{\pi^n}. \end{aligned}$$

This implies

$$\overline{\Delta}_1^{(8)} = \ell_1(2) - \ell_1(1) \mapsto w.$$

# The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

$$\begin{aligned} \ell_n(1) &\mapsto \frac{w^{2^n-1}}{\pi^n} \\ \ell_n(2) &\mapsto \frac{(\zeta_8 w)^{2^n-1}}{\pi^n}. \end{aligned}$$

This implies

$$\overline{\Delta}_1^{(8)} = \ell_1(2) - \ell_1(1) \mapsto w.$$

Thus we have shown that each factor of

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$$

and hence $D$ itself maps to a unit in $R_*$, thus proving the lemma.