A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theo braic Geometry
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1 Introduction

Introduction

The goal of this lecture is fourfold.

(i) To sketch part of the proof of the Slice Theorem.
(ii) To describe the spectrum $\tilde{\Omega}$ used to prove the main theorem.
(iii) To sketch the proof of the (yet to be stated) Periodicity Theorem.
(iv) To sketch the proof that the $\tilde{\Omega}^{C_8}$ and $\tilde{\Omega}^{hC_8}$ are equivalent, the Fixed Point Theorem.

Before we can do this, we need to introduce another concept from equivariant stable homotopy theory, that of geometric fixed points. 5.2

2 Geometric fixed points

Geometric fixed points

Unstably a G-space X has a fixed point set,

$$X^G = \{x \in X : \gamma(x) = x \, \forall \gamma \in G\}.$$

This is the same as $F(S^0, X_+)^G$, the space of based equivariant maps $S^0 \to X_+$, which is the same as the space of unbased equivariant maps $* \to X$.

The homotopy fixed point set X^{hG} is the space of based equivariant maps $EG_+ \to X_+$, where EG is a contractible free G-space. The equivariant homotopy type of X^{hG} is independent of the choice of EG. 5.3

Geometric fixed points (continued)

Both of these definitions have stable analogs, but the fixed point functor is awkward for two reasons: it fails to commute with smash products and with infinite suspensions.

The geometric fixed set $\Phi^G X$ is a convenient substitute that avoids these difficulties. In order to define it we need the isotropy separation sequence, which in the case of a finite cyclic 2-group G is

$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

Here EC_2 is a G-space via the projection $G \to C_2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a G-space. 5.4

Geometric fixed points (continued)

Under this action EC_2^G is empty while for any proper subgroup H of G, $EC_2^H = EC_2$, which is contractible. For an arbitrary finite group G it is possible to construct a G-space with the similar properties.

Definition. *For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is*

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

5.5

Geometric fixed points (continued)

We have the isotropy separation sequence

$$EC_{2+} \to S^0 \to \tilde{E}C_2$$

and

$$\Phi^G X := (X \wedge \tilde{E}C_2)^G.$$

This functor has the following properties:

- For G-spectra X and Y, $\Phi^G(X \wedge X) = \Phi^G X \wedge \Phi^G Y$.
- For a G-space X, $\Phi^G \Sigma^\infty X = \Sigma^\infty(X^G)$.
- A map $f : X \to Y$ is a G-equivalence iff $\Phi^H f$ is an ordinary equivalence for each subgroup $H \subset G$.

From the second property we can deduce that for $H \subset G$,

- $\Phi^H S^V = S^{V^H}$.
- $\Phi^H MU_{\mathbf{R}}^{((g/2))} = MO^{(g/h)}$, where MO is the unoriented cobordism spectrum.

5.6

Geometric fixed points (continued)

Geometric Fixed Point Theorem. *Let G be a finite cyclic 2-group and let $\overline{\rho}$ denote its reduced regular representation. Then for any G-spectrum X, $\pi_\star(\tilde{E}C_2 \wedge X) = a_{\overline{\rho}}^{-1}\pi_\star(X)$, where $a_{\overline{\rho}} \in \pi_{-\overline{\rho}}$ is the element defined in Lecture 4.*

5.7

To prove this will show that $E = \lim_{i\to\infty} S(i\overline{\rho})$ is G-equivalent to EC_2 by showing it has the appropriate fixed point sets. Since $(S(\overline{\rho}))^G$ is empty, the same is true of E^G. Since $(S(\overline{\rho}))^H$ for a proper subgroup H is $S^{|G/H|-2}$, its infinite join E^H is contractible.

It follows that $\tilde{E}C_2$ is equivalent to $\lim_{i\to\infty} S^{i\overline{\rho}}$, which implies the result.

Geometric fixed points (continued)

Recall that $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. In $\pi_{i\rho_g}(MU_{\mathbf{R}}^{((g/2))})$ we have the element

$$Nr_i = r_i(1)r_i(2)\cdots r_i(g/2).$$

Applying the functor Φ^G to the map $Nr_i : S^{i\rho_g} \to MU_{\mathbf{R}}^{((g/2))}$ gives a map $S^i \to MO$.

Lemma. *The generators r_i and y_i can be chosen so that*

$$\Phi^G Nr_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

5.8

3 The Slice Theorem

Toward the proof of the slice theorem

The Slice Theorem describes the slices associated with $MU_{\mathbf{R}}^{(g/2)}$. Its proof is a delicate induction argument. Here we will outline the proof of a key step in it.

Recall that

$$\pi_*^u(MU_{\mathbf{R}}^{(g/2)}) = \mathbf{Z}[r_i(j)\colon i>0,\ 1\le j\le g/2] \text{ with } |r_i(j)| = 2i.$$

There is a way to kill the $r_i(j)$ for any collection of is and get a new equivariant spectrum which is a module over the E_∞-ring spectrum $MU_{\mathbf{R}}^{(g/2)}$. We let $R_G(m)$ denote the result of killing the $r_i(j)$ for $i \le m$. 5.9

4 The Reduction Theorem

The Reduction Theorem

There are maps

$$MU_{\mathbf{R}}^{(g/2)} = R_G(0) \to R_G(1) \to R_G(2) \to \cdots \to H\mathbf{Z}$$

and we denote the limit by $R_G(\infty)$. A key step in the proof of the Slice Theorem is the following.

Reduction Theorem. *The map $f_G : R_G(\infty) \to H\mathbf{Z}$ is a weak G-equivalence.*

5.10

The nonequivariant analog of this statement is obvious. We will prove the corresponding statement over subgroups $H \subset G$ by induction on the order of H.

The Reduction Theorem (continued)

This means it suffices to show that $\Phi^H f$ is an ordinary equivalance for each subgroup $H \subset G$. To this end we will determine both $\pi_*(\Phi^H R_G(\infty))$ and $\pi_*(\Phi^H H\mathbf{Z})$.

As H-spectra we have $R_G(m) = R_H(m)^{(g/h)}$, so it suffices to determine $\pi_*(\Phi^G R_G(\infty))$. One can show that for each $m > 0$ there is a cofiber sequence

$$\Sigma^m \Phi^G R_G(m-1) \xrightarrow{\Phi^G N r_m} \Phi^G R_G(m-1) \longrightarrow \Phi^G R_G(m).$$

The lemma above determines the map $\Phi^G N r_m$. 5.11

The Reduction Theorem (continued)

We know that $\Phi^G R_G(0) = MO$ and $\Phi^G N r_1$ is trivial, so $\Phi^G R_G(1) = MO \wedge (S^0 \vee S^2)$.

Let $Q(m)$ denote the spectrum obtained from MO by killing the y_i for $i \le m$ so the limit $Q(\infty)$ is $H\mathbf{Z}/2$. Recall that y_i is not defined when $i = 2^k - 1$. Our cofiber sequence for $m = 2$ is the smash product of $S^0 \vee S^2$ with

$$\Sigma^2 MO \xrightarrow{y_2} MO \longrightarrow Q(2).$$

Similarly we find that $\Phi^G R_G(\infty) = \vee_{k\ge 0} \Sigma^{2k} H\mathbf{Z}/2$. 5.12

The spectrum $\Phi^G H\mathbf{Z}$

Recall that $\Phi^G H\mathbf{Z} = (\tilde{E}C_2 \wedge H\mathbf{Z})^H$. The action of the subgroup of index 2 is trivial, so this is the same as $(\tilde{E}C_2 \wedge H\mathbf{Z})^{C_2} = \Phi^{C_2} H\mathbf{Z}$.

Earlier we described the computation of

$$\pi_k(S^{m\rho_2} \wedge H\mathbf{Z}) = \pi_k(S^{m+m\sigma} \wedge H\mathbf{Z}) = \pi_{k-m-m\sigma}(H\mathbf{Z}).$$

This means we have all of $\pi_\star(H\mathbf{Z})$, the $RO(C_2)$-graded homotopy of $H\mathbf{Z}$. It turns out that $a_\sigma^{-1}\pi_\star(H\mathbf{Z}) = \mathbf{Z}/2[u_{2\sigma}, a_\sigma^{\pm 1}]$, where $u_{2\sigma} \in \pi_{2-2\sigma}$. The integrally graded part of this is $\mathbf{Z}/2[b]$ where $b = u_{2\sigma}/a_\sigma^2 \in \pi_2$.

Hence $\pi_*(\Phi^G H\mathbf{Z})$ and $\pi_*(\Phi^G R_G(\infty))$ are abstractly isomorphic. A more careful analysis shows that f induces this isomorphism, thereby proving the Reduction Theorem. 5.13

5 The Periodicity Theorem

Some differentials in the slice spectral sequence

Before we can state and prove the Periodicity Theorem, we need to explore some differentials in the slice spectral sequence for $MU_{\mathbf{R}}^{(g/2)}$.

It follows from the Slice and Vanishing Theorems that it has a vanishing line of slope $g-1$. The only slice cells which reach this line are the ones not induced from a proper subgroup, namely the $S^{n\rho_g}$ associated with the subring $\mathbf{Z}[Nr_i : i > 0]$.

For each $i > 0$ there is an element

$$f_i \in \pi_i(S^{i\rho_g}) \subset E_2^{(g-1)i, gi},$$

the bottom element in $\pi_*(S^{i\rho_g} \wedge H\mathbf{Z})$. 5.14

Some slice differentials (continued)

It is the composite $S^i \xrightarrow{a_{i\rho_g}} S^{i\rho_g} \xrightarrow{Nr_i} MU_{\mathbf{R}}^{(g/2)}$.

The subring of elements on the vanishing line is $\mathbf{Z}[f_i : i > 0]/(2f_i)$. Under the map

$$\pi_*(MU_{\mathbf{R}}^{(g/2)}) \to \pi_*(\Phi^G MU_{\mathbf{R}}^{(g/2)}) = \pi_*(MO)$$

we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$

It follows that any differentials hitting the vanishing line must land in the ideal $(f_1, f_3, f_7, \dots)$. A similar statement can be made after smashing with $S^{2^k\sigma}$. 5.15

Some slice differentials (continued)

Slice Differentials Theorem. *In the slice spectral sequence for $\Sigma^{2^k\sigma} MU_{\mathbf{R}}^{(g/2)}$ (for $k > 0$) we have $d_r(u_{2^k\sigma}) = 0$ for $r < 1 + (2^k - 1)g$, and*

$$d_{1+(2^k-1)g}(u_{2^k\sigma}) = a_\sigma^{2^k} f_{2^k-1}.$$

5.16

Inverting a_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$. This means each f_{2^k-1} must be killed by some power of a_σ. The only way this can happen is as indicated in the theorem.

Some slice differentials (continued)

Let

$$\overline{\Delta}_k^{(g)} = N_2^g r_{2^k-1} \in \pi_{(2^k-1)\rho_g}(MU_{\mathbf{R}}^{(g/2)}).$$

We want to invert this element and study the resulting slice spectral sequence. As explained previously, it is confined to the first and third quadrants with vanishing lines of slopes 0 and $g-1$.

The differential d_r on $u_{2^{k+1}\sigma}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{\Delta}_k^{(g)}$, then $u_{2^{k+1}\sigma}$ will be a permanent cycle. 5.17

Some slice differentials (continued)

We have

$$\begin{aligned} f_{2^{k+1}-1}\overline{\Delta}_k^{(g)} &= a_{(2^{k+1}-1)\rho_g} N_2^g r_{2^{k+1}-1} N_2^g r_{2^k-1} \\ &= a_{2^k\rho_g} \overline{\Delta}_{k+1}^{(g)} f_{2^k-1} \\ &= \overline{\Delta}_{k+1}^{(g)} d_{r'}(u_{2^k\sigma}) \text{ for } r' < r. \end{aligned}$$

Corollary. *sequence for* $\left(\overline{\Delta}_k^{(g)}\right)^{-1} MU_{\mathbf{R}}^{(g/2)}$*, the class* $u_{2\sigma}^{2^k}$ *is a permanent cycle.* 5.18

The Periodicity Theorem

The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle. We need a similar statement about a power of $u_{2\rho_g}$ when $g = 2^n$.

We will get this by using the norm property of u, namely that if W is an oriented representation of a subgroup $H \subset G$ with $W^H = 0$ and induced representation W', then the norm functor N_h^g from H-spectra to G-spectra satisfies $N_H^G(u_W) u_{2\rho_{G/H}}^{|W|/2} = u_{W'}$.

From this we can deduce that $u_{2\rho_g} = \prod_{m=1}^n N_{2^m}^{2^n}(u_{2^m\sigma_m})$, where σ_m denotes the sign representation on C_{2^m}. 5.19

The Periodicity Theorem (continued)

In particular we have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\Delta}_{k_m}^{(2^m)}$ for $1 \le m \le 3$. If we make k_m too small we will lose the detection property, that is we will get a spectrum that does not detect the θ_j. It turns out that k_m must be chosen so that $8|2^m k_m$. This will be explained in the last lecture.

- Inverting $\overline{\Delta}_4^{(2)}$ makes $u_{32\sigma_1}$ a permanent cycle.
- Inverting $\overline{\Delta}_2^{(4)}$ makes $u_{8\sigma_2}$ a permanent cycle.
- Inverting $\overline{\Delta}_1^{(8)}$ makes $u_{4\sigma_3}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8}$ a permanent cycle. 5.20

The Periodicity Theorem (continued)

Let

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)}).$$

The we define $\tilde{\Omega} = D^{-1} MU_{\mathbf{R}}^{(4)}$ and $\Omega = \tilde{\Omega}^{C_8}$.

Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(\Omega)$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0. 5.21

The Periodicity Theorem (continued)

Preperiodicity Theorem. *Let* $\Delta_1^{(8)} = u_{2\rho_8}(\overline{\Delta}_1^{(8)})^2 \in E_2^{16,0}(D^{-1}MU_{\mathbf{R}}^{(4)})$*. Then* $(\Delta_1^{(8)})^{16}$ *is a permanent cycle.*

5.22

To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^{32}$. Both $u_{32\rho_8}$ and $\overline{\Delta}_1^{(8)}$ are permanent cycles, so $(\Delta_1^{(8)})^{16}$ is also one.

Thus we have an equivariant map $\Sigma^{256}D^{-1}MU_{\mathbf{R}}^{(4)} \to D^{-1}MU_{\mathbf{R}}^{(4)}$ and a similar map on the fixed point set. The latter one is invertible because $u_{2\rho_8}^{32}$ restricts to the identity.

Thus we have proved

Periodicity Theorem. *Let* $\Omega = (D^{-1}MU_{\mathbf{R}}^{(4)})^{C_8}$*. Then* $\Sigma^{256}\Omega$ *is equivalent to* Ω*.*

6 The Fixed Point Theorem

The Fixed Point Theorem

In order to finish the proof of the main theorem, we need to show that the actual fixed point set of $\tilde{\Omega} = D^{-1}MU_{\mathbf{R}}^{(4)}$ is equivalent to the homotopy fixed point set. We call this statement the Fixed Point Theorem.

The slice spectral sequence computes the homotopy of the former while the Hopkins-Miller spectral sequence (which is known to detect θ_j) computes that of the latter. 5.23

The Fixed Point Theorem (continued)

Here is a general approach to showing that actual and homotopy fixed points are equivalent for a G-spectrum X.

We have an equivariant map $EG_+ \to S^0$. Mapping both into X gives a map of G-spectra $\varphi : X+ \to F(EG_+, X_+)$. Passing to fixed points would give a map $X^G \to X^{hG}$, but we will prove the stronger statement that φ is a G-equivalence.

The case of interest is $X = \tilde{\Omega}$ and $G = C_8$. We will argue by induction on the order of the subgroups H of G, the statement being obvious for the trivial group. We will smash φ with the isotropy separation sequence

$$EG_+ \to S^0 \to \tilde{E}G.$$

5.24

The Fixed Point Theorem (continued)

This gives us the following diagram in which both rows are cofiber sequences.

$$\begin{array}{ccccc}
EG_+ \wedge \tilde{\Omega} & \longrightarrow & \tilde{\Omega} & \longrightarrow & \tilde{E}G \wedge \tilde{\Omega} \\
\downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' \\
EG_+ \wedge F(EG_+, \tilde{\Omega}) & \longrightarrow & F(EG_+, \tilde{\Omega}) & \longrightarrow & \tilde{E}G \wedge F(EG_+, \tilde{\Omega})
\end{array}$$

The map φ' is an equivalence because $\tilde{\Omega}$ is nonequivariantly equivalent to $F(EG_+, \tilde{\Omega})$, and EG_+ is built up entirely of free G-cells.

Thus it suffices to show that φ'' is an equivalence, which we will do by showing that both its source and target are contractible. Both have the form $\tilde{E}G \wedge X$ where X is a module spectrum over $\tilde{\Omega}$, so it suffices to show that $\tilde{E}G \wedge \tilde{\Omega}$ is contractible. 5.25

The Fixed Point Theorem (continued)

We need to show that $\tilde{E}G \wedge \tilde{\Omega}$ is G-equivariantly contractible. We will show that it is H-equivariantly contractible by induction on the order of the subgroups H of G. Over the trivial group $\tilde{E}G$ itself is contractible. Let H be a subgroup, $H' \subset H$ the subgroup of index 2 and $H_2 = H/H'$.

We will smash our spectrum with the cofiber sequence

$$EH_{2+} \to S^0 \to \tilde{E}H_2.$$

Then $\tilde{E}H_2 \wedge \tilde{E}G \wedge \tilde{\Omega}$ is contractible over H', so it suffices to show that it H-fixed point set is contractible. It is

$$\Phi^H(\tilde{E}G \wedge \tilde{\Omega}) = \Phi^H(\tilde{E}G) \wedge \Phi^H(\tilde{\Omega}),$$

and $\Phi^H(\tilde{\Omega})$ is contractible because $\Phi^H(D) = 0$.

Thus it remains to show that $EH_{2+} \wedge \tilde{E}G \wedge \tilde{\Omega}$ is H-contractible. But this is equivalent to the H'-contractibility of $\tilde{E}G \wedge \tilde{\Omega}$, which we have by induction. 5.26