Lecture 5: The Reduction, Periodicity and Fixed Point Theorems

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
Tokyo City University

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Introduction

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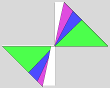

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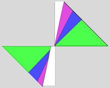

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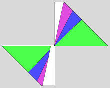
(ii) To describe the spectrum $\tilde{\Omega}$ used to prove the main theorem.



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Before we can do this, we need to introduce another concept from equivariant stable homotopy theory, that of geometric fixed points.

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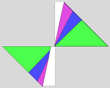

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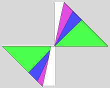

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Here EC_2 is a G-space via the projection $G \to C_2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a G-space.

Geometric fixed points (continued)

Under this action EC_2^G is empty while for any proper subgroup H of G, $EC_2^H = EC_2$, which is contractible. For an arbitrary finite group G it is possible to construct a G-space with the similar properties.

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Definition

For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

Geometric fixed points (continued)

We have the isotropy separation sequence

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From the second property we can deduce that for $H \subset G$,

- $\Phi^H S^V = S^{V^H}$.
- $\Phi^H MU_{\mathbf{R}}^{((g/2)} = MO^{(g/h)}$, where MO is the unoriented cobordism spectrum.

Geometric fixed points (continued)

Geometric Fixed Point Theorem

Let G be a finite cyclic 2-group and let $\overline{\rho}$ denote its reduced regular representation. Then for any G-spectrum X, $\pi_\star(\tilde{E}C_2 \wedge X) = a_{\overline{\rho}}^{-1}\pi_\star(X)$, where $a_{\overline{\rho}} \in \pi_{-\overline{\rho}}$ is the element defined in Lecture 4.

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It follows that $\tilde{E}C_2$ is equivalent to $\lim_{i\to\infty} S^{i\,\overline{\rho}}$, which implies the result.

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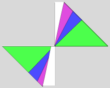
Recall that $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$.



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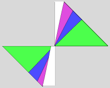

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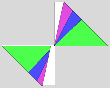
Lemma

The generators r_i and y_i can be chosen so that

$$\Phi^G Nr_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

Toward the proof of the slice theorem

The Slice Theorem describes the slices associated with $MU_{\mathbf{R}}^{(g/2)}$. Its proof is a delicate induction argument. Here we will outline the proof of a key step in it.



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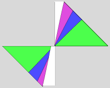

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There is a way to kill the $r_i(j)$ for any collection of is and get a new equivariant spectrum which is a module over the E_∞-ring spectrum $MU_{\mathbf{R}}^{(g/2)}$. We let $R_G(m)$ denote the result of killing the $r_i(j)$ for $i \leq m$.

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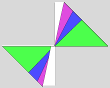
Reduction Theorem

The map $f_G : R_G(\infty) \to H\mathbf{Z}$ is a weak G-equivalence.

The nonequivariant analog of this statement is obvious. We will prove the corresponding statement over subgroups $H \subset G$ by induction on the order of H.



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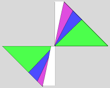

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$$\Sigma^m \Phi^G R_G(m-1) \xrightarrow{\Phi^G Nr_m} \Phi^G R_G(m-1) \longrightarrow \Phi^G R_G(m).$$

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The lemma above determines the map $\Phi^G N r_m$.

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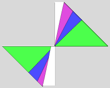

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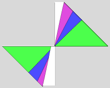

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Similarly we find that $\Phi^G R_G(\infty) = \vee_{k \geq 0} \Sigma^{2k} H\mathbf{Z}/2$.

The spectrum $\Phi^G H\mathbf{Z}$

Recall that $\Phi^G H\mathbf{Z} = (\tilde{E}C_2 \wedge H\mathbf{Z})^H$.

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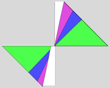
Hence $\pi_*(\Phi^G H\mathbf{Z})$ and $\pi_*(\Phi^G R_G(\infty))$ are abstractly isomorphic.



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Hence $\pi_*(\Phi^G H\mathbf{Z})$ and $\pi_*(\Phi^G R_G(\infty))$ are abstractly isomorphic. A more careful analysis shows that f induces this isomorphism, thereby proving the Reduction Theorem.

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Some differentials in the slice spectral sequence

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For each $i > 0$ there is an element

$$f_i \in \pi_i(S^{i\rho_g}) \subset E_2^{(g-1)i,gi},$$

the bottom element in $\pi_*(S^{i\rho_g} \wedge H\mathbf{Z})$.

Some slice differentials (continued)

It is the composite $S^i \xrightarrow{a_{i\rho_g}} S^{i\rho_g} \xrightarrow{Nr_i} MU_{\mathbf{R}}^{(g/2)}$.

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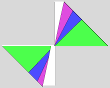
It is the composite $S^{i} \xrightarrow{a_{i\rho_g}} S^{i\rho_g} \xrightarrow{Nr_i} MU_{\mathbf{R}}^{(g/2)}$.
The subring of elements on the vanishing line is $\mathbf{Z}[f_i : i > 0]/(2f_i)$.



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The subring of elements on the vanishing line is $\mathbf{Z}[f_i : i > 0]/(2f_i)$. Under the map

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we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$

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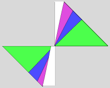

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It follows that any differentials hitting the vanishing line must land in the ideal $(f_1, f_3, f_7, \ldots)$. A similar statement can be made after smashing with $S^{2^k\sigma}$.

Some slice differentials (continued)

Slice Differentials Theorem

In the slice spectral sequence for $\Sigma^{2^k\sigma} MU_{\mathbf{R}}^{((g/2))}$ *(for* $k > 0$*) we have* $d_r(u_{2^k\sigma}) = 0$ *for* $r < 1 + (2^k - 1)g$*, and*

$$d_{1+(2^k-1)g}(u_{2^k\sigma}) = a_\sigma^{2^k} f_{2^k-1}.$$

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Inverting a_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$.

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Some slice differentials (continued)

Let

$$\overline{\Delta}_k^{(g)} = N_2^g r_{2^k-1} \in \pi_{(2^k-1)\rho_g}(MU_{\mathbf{R}}^{(g/2)}).$$

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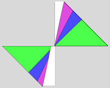
We want to invert this element and study the resulting slice spectral sequence.



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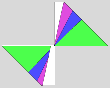
The differential d_r on $u_{2^{k+1}\sigma}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line.



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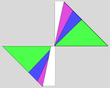
The differential d_r on $u_{2^{k+1}\sigma}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{\Delta}_k^{(g)}$, then $u_{2^{k+1}\sigma}$ will be a permanent cycle.

Some slice differentials (continued)

We have

$$f_{2^{k+1}-1}\overline{\Delta}_k^{(g)} \quad = \quad a_{(2^{k+1}-1)\rho_g} N_2^g r_{2^{k+1}-1} N_2^g r_{2^k-1}$$



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$$

Corollary

sequence for $\left(\overline{\Delta}_k^{(g)}\right)^{-1} MU_{\mathbf{R}}^{(g/2)}$*, the class* $u_{2\sigma}^{2^k}$ *is a permanent cycle.*

The Periodicity Theorem

The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle.

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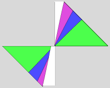

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From this we can deduce that $u_{2\rho_g} = \prod_{m=1}^{n} N_{2^m}^{2^n}(u_{2^m\sigma_m})$,

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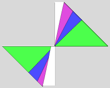
The Periodicity Theorem (continued)

In particular we have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.



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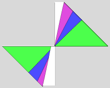

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The Periodicity Theorem (continued)

In particular we have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\Delta}_{k_m}^{(2^m)}$ for $1 \le m \le 3$. If we make k_m too small we will lose the detection property, that is we will get a spectrum that does not detect the θ_j. It turns out that k_m must be chosen so that $8 | 2^m k_m$. This will be explained in the last lecture.

- Inverting $\overline{\Delta}_4^{(2)}$ makes $u_{32\sigma_1}$ a permanent cycle.
- Inverting $\overline{\Delta}_2^{(4)}$ makes $u_{8\sigma_2}$ a permanent cycle.
- Inverting $\overline{\Delta}_1^{(8)}$ makes $u_{4\sigma_3}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8}$ a permanent cycle.

The Periodicity Theorem (continued)

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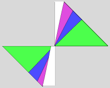
Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(\Omega)$ has the usual properties:



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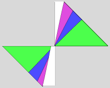
- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0.



The Periodicity Theorem (continued)

Preperiodicity Theorem

Let $\Delta_1^{(8)} = u_{2\rho_8}(\overline{\Delta}_1^{(8)})^2 \in E_2^{16,0}(D^{-1}MU_{\mathbf{R}}^{(4)})$*. Then* $(\Delta_1^{(8)})^{16}$ *is a permanent cycle.*

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The Periodicity Theorem (continued)

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To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^{32}$.

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Thus we have an equivariant map $\Sigma^{256}D^{-1}MU_{\mathbf{R}}^{(4)} \to D^{-1}MU_{\mathbf{R}}^{(4)}$ and a similar map on the fixed point set. The latter one is invertible because $u_{2\rho_8}^{32}$ restricts to the identity.

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Periodicity Theorem

Let $\Omega = (D^{-1}MU_{\mathbf{R}}^{((4))})^{C_8}$*. Then* $\Sigma^{256}\Omega$ *is equivalent to* Ω*.*

The Fixed Point Theorem

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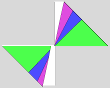
The slice spectral sequence computes the homotopy of the former while the Hopkins-Miller spectral sequence (which is known to detect θ_j) computes that of the latter.

The Fixed Point Theorem (continued)

Here is a general approach to showing that actual and homotopy fixed points are equivalent for a G-spectrum X.



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The case of interest is $X = \tilde{\Omega}$ and $G = C_8$. We will argue by induction on the order of the subgroups H of G, the statement being obvious for the trivial group. We will smash φ with the isotropy separation sequence

$$EG_+ \to S^0 \to \tilde{E}G.$$

The Fixed Point Theorem (continued)

This gives us the following diagram in which both rows are cofiber sequences.

$$\begin{tikzcd}
EG_+ \wedge \tilde{\Omega} \arrow[r] \arrow[d, "\varphi'"] & \tilde{\Omega} \arrow[r] \arrow[d, "\varphi"] & \tilde{E}G \wedge \tilde{\Omega} \arrow[d, "\varphi''"] \\
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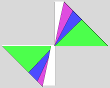

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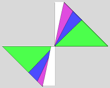

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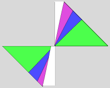

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Thus it remains to show that $EH_{2+} \wedge \tilde{E}G \wedge \tilde{\Omega}$ is H-contractible. But this is equivalent to the H'-contractibility of $\tilde{E}G \wedge \tilde{\Omega}$, which we have by induction.