A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theo ... braic Geometry
Tokyo City University

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Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester 3.1

Our strategy

Recall our goal is to prove

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$. 3.2

Our strategy is to find a map $S^0 \to \Omega$ to a nonconnective spectrum Ω with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial.
(ii) It is 256-periodic, meaning $\Sigma^{256}\Omega \cong \Omega$.
(iii) $\pi_{-2}(\Omega) = 0$.

1 Equivariant stable homotopy theory

1.1 G-spaces

G-spaces

Before we can describe any of this, we need to introduce equivariant stable homotopy theory.


Peter May


John Greenlees


Gaunce Lewis

Let G be a finite group. A G-space is a topological space X with a continuous left action by G; a based G-space is a G-space together with a basepoint fixed by G.

We can convert an unbased G-spaces X into a based one by taking the topological sum of X and a G-fixed basepoint, denoted by X_+. 3.3

Products and maps of G-spaces

The product $X \times Y$ of two G-spaces is a G-space under the diagonal action, as is the smash product of two based G-spaces.

The space $F(X,Y)$ of based maps $X \to Y$ is itself a G-space with G-action defined by $(\gamma f)(x) = \gamma f(\gamma^{-1} x)$ for $\gamma \in G$.

Its fixed point set $F(X,Y)^G$ is the space of based G-maps $X \to Y$, i.e., those maps commuting with the action of G.

We use the notation $[X,Y]_G$ to denote the set of homotopy classes of based G-maps $X \to Y$.

A map of G-spaces $f : X \to Y$ is said to be a weak G-equivalence if for each subgroup $H \subset G$, the induced map $f : X^H \to Y^H$ is a weak equivalence in the nonequivariant sense. 3.4

1.2 G-CW complexes

G-CW complexes via orbits

There are two ways to generalize the construction of CW-complexes to the equivariant world, one using orbits and one using representations.

For the orbit construction, given any subgroup H of G we may form the homogeneous space G/H and its based counterpart, G/H_+.

These are treated as 0-dimensional cells, and they play a role in equivariant theory analogous to the role of points in nonequivariant theory. 3.5

G-CW complexes via orbits (continued)

We form the n-dimensional cells from these homogeneous spaces. In the unbased context, the cell-sphere pair is

$$(G/H \times D^n, G/H \times S^{n-1})$$

and in the based context

$$(G/H_+ \wedge D^n, G/H_+ \wedge S^{n-1}).$$

A cell is said to be induced if it comes from a proper subgroup H.

Starting from these cell-sphere pairs, we form G-CW complexes exactly as nonequivariant CW-complexes are formed from the cell-sphere pairs (D^n, S^{n-1}). In such a complex, an element $\gamma \in G$ acts on a cell either by mapping it homeomorphically to another cell or by fixing it. 3.6

G-CW complexes via representations

Let V be an orthogonal representation of G. Denote its one-point compactification by S^V, with ∞ as the basepoint. We denote the trivial n-dimensional real representation by n, giving the symbol S^n its usual meaning.

We may also form the unit disc and unit sphere

$$D(V) = \{v \in V : ||v|| \leq 1\} \text{ and } S(V) = \{v \in V : ||v|| = 1\};$$

we think of them as unbased G-spaces. There is a homeomorphism $S^V \cong D(V)/S(V)$.

We can use these objects to build G-CW complexes as well. In this case G can act on an individual cell by "rotating" it via the representation V. 3.7

More general G-CW complexes

We can also mix these two constructions by considering cell-sphere pairs such as

$$(G \times_H D(V), G \times_H S(V))$$

and

$$(G_+ \wedge_H D(V), G_+ \wedge_H S(V)),$$

where V is a representation of the subgroup H.

In such a complex, individual cells may be either permuted or rotated by an element of G. 3.8

1.3 Ordinary spectra

Toward equivariant spectra

Before defining equivariant spectra, we need to recall the definition of an ordinary spectrum.

A prespectrum D is a collection of spaces $\{D_n : n \gg 0\}$ with maps $\Sigma D_n \to D_{n+1}$. The adjoint of the structure map is a map $D_n \to \Omega D_{n+1}$.

We get a spectrum $E = \{E_n : n \in \mathbf{Z}\}$ from the prespectrum D by defining

$$E_n = \varinjlim_k \Omega^k D_{n+k}$$

This makes E_n homeomorphic to ΩE_{n+1}.

For technical reasons it is convenient to replace the collection $\{E_n\}$ by $\{E_V\}$ indexed by finite dimensional subspaces V of a countably infinite dimensional real vector space $\mathscr{U}$ called a universe. 3.9

Toward equivariant spectra (continued)

The homotopy type of E_V depends only on the dimension of V and there are homeomorphisms

$$E_V \to \Omega^{|W|-|V|} E_W \qquad \text{for } V \subset W \subset \mathscr{U}.$$

A map of spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps. 3.10

1.4 Equivariant spectra

G-equivariant spectra

Let G be a finite group. Experience has shown that in order to do equivariant stable homotopy theory, one needs G-spaces E_V indexed by finite dimensional orthogonal representations V sitting in a countably infinite dimensional orthogonal representation $\mathscr{U}$.

This universe $\mathscr{U}$ is said to be complete if it contains infinitely many copies of each irreducible representation of G. A canonical example of a complete universe for finite G is the direct sum of countably many copies of the regular real representation of G. 3.11

G-equivariant spectra (continued)

A G-equivariant spectrum (G-spectrum for short) indexed on $\mathscr{U}$ consists of a based G-space E_V for each finite dimensional subspace $V \subset \mathscr{U}$ together with a transitive system of based G-homeomorphisms

$$E_V \xrightarrow[\cong]{\tilde{\sigma}_{V,W}} \Omega^{W-V} E_W$$

for $V \subset W \subset \mathscr{U}$. Here $\Omega^V X = F(S^V, X)$ and $W - V$ is the orthogonal complement of V in W. As in the classical case, the G-homotopy type of E_V depends only on the isomorphism class of V. 3.12

G-equivariant spectra (continued)

A map of G-spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.

Dropping the requirement that the structure maps be homeomorphisms gives us a G-prespectrum.

The structure map $\tilde{\sigma}_{V,W}$ is adjoint to a map

$$\sigma_{V,W} : \Sigma^{W-V} E_V \to E_W,$$

where $\Sigma^V X$ is defined to be $S^V \wedge X$.

A suspension G-prespectrum is a G-prespectrum in which the maps above are G-equivalences for V sufficiently large. 3.13

1.5 $RO(G)$-graded homotopy

$RO(G)$-graded homotopy groups

Given a representation V one has a suspension G-spectrum $\Sigma^\infty S^V$, which is often denoted abusively (as in the nonequivariant case) by S^V.

As in the nonequivariant case, to define a prespectrum D it suffices to define G-spaces DV for a cofinal collection of representations V.

We define S^{-V} by saying its Wth space for $V \subset W$ is S^{W-V}. This is the analog of formal desuspension in the nonequivariant case. 3.14

$RO(G)$-graded homotopy groups (continued)

Given a virtual representation $\nu = W - V$, we define $S^\nu = \Sigma^W S^{-V}$. Hence we have a collection of sphere spectra graded over the orthogonal representation ring $RO(G)$.

We define

$$\pi_\nu^G(X) = [S^\nu, X]_G,$$

the $RO(G)$-graded homotopy groups of the G-spectrum X. 3.15

2 MU as a C_2-spectrum

MU as a C_2-spectrum

Let ρ denote the real regular representation of C_2. It is isomorphic to the complex numbers $\mathbf{C}$ with conjugation.

We define a C_2-prespectrum mu by $mu_{k\rho} = MU(k)$, the Thom space of the universal $\mathbf{C}^k$-bundle over $BU(k)$, which is a direct limit of complex Grassmannian manifolds. The action of C_2 is by complex conjugation.

Since any orthogonal representation V of C_2 is contained in $k\rho$ for $k \gg 0$, we can define the C_2-spectrum MU by

$$MU_V = \lim_{\substack{\to \\ k}} \Omega^{k\rho - V} MU(k).$$

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MU as a C_2-spectrum (continued)

This spectrum in known as real cobordism theory $MU_{\mathbf{R}}$ and has been studied by Landweber, Araki, Hu-Kriz and Kitchloo-Wilson.


Peter Landweber


Shoro Araki 1930–2005


Nitu Kitchloo


Steve Wilson


Igor Kriz and Po Hu 3.17

3 The norm functor

Inducing and coinducing up to a larger group

Let $H \subset G$ be groups and let X be a H-space. There are two ways to get a G-space from it. The corresponding functors are the left and right adjoints to the forgetful functor from G-spaces to H-spaces.

There is the induced G-space $G \times_H X$. Its underlying space is the disjoint union of $|G/H|$ copies of X.

An example is the the cell-sphere pair

$$(G/H \times D^n, G/H \times S^{n-1}).$$ 3.18

Inducing and coinducing up to a larger group (continued)

There is also the coinduced G-space

$$\begin{aligned} \mathrm{map}_H(G,X) &= \{f \in \mathrm{map}(G,X)\colon f(\gamma\eta^{-1}) = \eta f(\gamma) \\ &\qquad \forall \eta \in H \text{ and } \gamma \in G\} \end{aligned}$$

The underlying space here is the Cartesian product $X^{|G/H|}$.

There is a based analog of the coinduced G-space in which the underlying space is the smash product $X^{(|G/H|)}$.

It extends to H-spectra. For a H-spectrum X we denote the coinduced G-spectrum by $N_H^G X$, the norm of X along the inclusion $H \subset G$. 3.19

Norming up from MU

We apply this construction to the case $H = C_2$, $G = C_{2^{n+1}}$ and $X = MU_{\mathbf{R}}$. The underlying spectrum of $N_H^G MU_{\mathbf{R}}$ is the 2^n-fold smash power $MU^{(2^n)}$.

Let $\gamma \in G$ be a generator and let z_i be a point in $MU_{\mathbf{R}}$. Then the action of G on $MU_{\mathbf{R}}^{(2^n)}$ is given by

$$\gamma(z_1 \wedge \cdots \wedge z_{2^n}) = \overline{z}_{2^n} \wedge z_1 \wedge \cdots \wedge z_{2^n-1},$$

where $\overline{z}_{2^n}$ is the complex conjugate of z_{2^n}. 3.20

4 Our spectrum Ω

Our spectrum Ω

In particular this makes $MU_{\mathbf{R}}^{(4)}$ into a C_8-spectrum. Our spectrum $\tilde{\Omega}$ is obtained from it by equivariantly inverting a certain element in its homotopy. Then $\Omega = \tilde{\Omega}^{C_8}$, which we will show to be equivalent to $\tilde{\Omega}^{hC_8}$.

The spectrum $MU_{\mathbf{R}}^{(4)}$ has two advantages over our earlier candidate E_4.

(i) It is a C_8-equivariant spectrum, while E_4 was merely an ordinary spectrum with a C_8 "action" for which a homotopy fixed point set could be defined.

(ii) The action of C_8 on $\pi_*(MU_{\mathbf{R}}^{(4)})$ is transparent, unlike its mysterious action on $\pi_*(E_4)$. 3.21

Our strategy (continued)

Our spectrum Ω will be derived from $MU_{\mathbf{R}}^{(4)}$ regarded as a C_8-spectrum.

We need to describe the homotopy of the underlying nonequivariant spectrum, which we denote $\pi_*^u(MU_{\mathbf{R}}^{(4)})$. 3.22

5 $\pi_*^u(MU_{\mathbf{R}}^{(4)})$

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$

Recall that $H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$. b_i is the image of a suitable generator of $H_{2i}(\mathbf{C}P^\infty)$ under the map

$$\Sigma^{\infty-2}\mathbf{C}P^\infty = \Sigma^{\infty-2}MU(1) \to MU.$$

It follows that $H_*(MU_{\mathbf{R}}^{(4)})$ is the 4-fold tensor power of this polynomial algebra. We denote its generators by $b_i(j)$ for $1 \le j \le 4$.

The action of γ on these generators is given by

$$\gamma(b_i(j)) = \begin{cases} b_i(j+1) & \text{for } 1 \le j \le 3 \\ (-1)^i b_i(1) & \text{for } j = 4. \end{cases}$$

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$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ is also a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $r_i(j)$ for $1 \le j \le 4$. The action of $G = C_8$ is similar to that on the $b_i(j)$, namely

$$\gamma(r_i(j)) = \begin{cases} r_i(j+1) & \text{for } 1 \le j \le 3 \\ (-1)^i r_i(1) & \text{for } j = 4. \end{cases}$$

Earlier we said that $\pi_*(MU_{\mathbf{R}}) = \mathbf{Z}[x_i\colon i > 0]$ with $|x_i| = 2i$. We are using different notation now because $r_i(j)$ need not be the image of x_i under any map $MU_{\mathbf{R}} \to MU_{\mathbf{R}}^{(4)}$. 3.24

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

Here is some useful notation. For a subgroup $H \subset G$, let $h = |H|$, let ρ_h denote its regular real representation and for $m \in \mathbf{Z}$, let

$$\widehat{S}(m\rho_h) = G_+ \wedge_H S^{m\rho_h}.$$

The underlying spectrum here is a wedge of $|G/H|$ copies of S^{mh}. We call this a slice cell of dimension mh.

We will explain how $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ is related to maps from the $\widehat{S}(m\rho_h)$. The following notion is helpful.

Definition. *Suppose X is a G-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map*

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension k whose underlying spheres represent a basis of $\pi_k^u(X)$.

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$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

Recall that in $\pi_*^u(MU_{\mathbf{R}})$, any monomial in the polynomial generators in dimension $2m$ is represented by an equivariant map from $S^{m\rho_2}$.

$\pi_*^u(MU^{(4)})$ is a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $r_i(j)$ for $1 \leq j \leq 4$. The action of a generator $\gamma \in G = C_8$ is given by

$$\gamma(r_i(j)) = \begin{cases} r_i(j+1) & \text{for } 1 \leq j \leq 3 \\ (-1)^i r_i(1) & \text{for } j = 4. \end{cases}$$

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$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

We will explain how $\pi_*^u(MU^{(4)})$ can be refined.

$\pi_2^u(MU^{(4)})$ has 4 generators $r_1(j)$ that are permuted up to sign by G. It is refined by an equivariant map

$$\widehat{W}_1 = \widehat{S}(\rho_2) \to MU^{(4)}.$$

Recall that the underlying spectrum of $\widehat{W}_1$ is a wedge of 4 copies of S^2.

3.27

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

In $\pi_4^u(MU^{(4)})$ there are 14 monomials that fall into 4 orbits under the action of G, each corresponding to a map from a slice cell.

$$\begin{array}{rcl} \widehat{S}(2\rho_2) & \longleftrightarrow & \{r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2\} \\ \widehat{S}(2\rho_2) & \longleftrightarrow & \{r_1(1)r_1(2), r_1(2)r_1(3), r_1(3)r_1(4), r_1(4)r_1(1)\} \\ \widehat{S}(2\rho_2) & \longleftrightarrow & \{r_2(1), r_2(2), r_2(3), r_2(4)\} \\ \widehat{S}(\rho_4) & \longleftrightarrow & \{r_1(1)r_1(3), r_1(2)r_1(4)\} \end{array}$$

(Recall that $\widehat{S}(\rho_4)$ is underlain by $S^4 \vee S^4$.) It follows that $\pi_4^u(MU^{(4)})$ is refined by an equivariant map from

$$\widehat{W}_2 = \widehat{S}(2\rho_2) \vee \widehat{S}(2\rho_2) \vee \widehat{S}(\rho_4) \vee \widehat{S}(2\rho_2).$$

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The refinement of $\pi_*^u(MU^{(4)})$ (continued)

A similar analysis can be made in any even dimension. G always permutes monomials up to sign. The first case of a singleton orbit occurs in dimension 8, namely

$$\widehat{S}(\rho_8) \quad \longleftrightarrow \quad \{r_1(1)r_1(2)r_1(3)r_1(4)\}.$$

Each group $\pi_{2n}^u(MU_{\mathbf{R}}^{(4)})$ can be refined by a map from a wedge of slice cells $\widehat{W}_n$. Note that $\widehat{S}(m\rho_1)$ never occurs as a wedge summand of $\widehat{W}_n$ because no monomial has a free orbit. 3.29