Lecture 3: Equivariant stable homotopy theory

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
Tokyo City University

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Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester

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Recall our goal is to prove

Main Theorem

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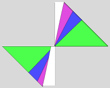
(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial.



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(iii) $\pi_{-2}(\Omega) = 0$.

G-spaces

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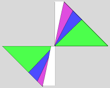

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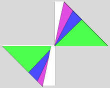

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We can convert an unbased G-spaces X into a based one by taking the topological sum of X and a G-fixed basepoint, denoted by X_+.

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A map of G-spaces $f : X \to Y$ is said to be a weak G-equivalence if for each subgroup $H \subset G$, the induced map $f : X^H \to Y^H$ is a weak equivalence in the nonequivariant sense.

G-CW complexes via orbits

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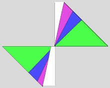

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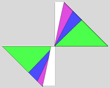
For the orbit construction, given any subgroup H of G we may form the homogeneous space G/H and its based counterpart, G/H_+.

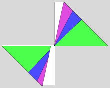
These are treated as 0-dimensional cells, and they play a role in equivariant theory analogous to the role of points in nonequivariant theory.



G-CW complexes via orbits (continued)

We form the n-dimensional cells from these homogeneous spaces. In the unbased context, the cell-sphere pair is

$$(G/H \times D^n, G/H \times S^{n-1})$$



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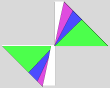

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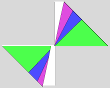

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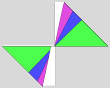

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We can use these objects to build G-CW complexes as well. In this case G can act on an individual cell by “rotating” it via the representation V.

More general G-CW complexes

We can also mix these two constructions by considering cell-sphere pairs such as

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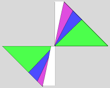
For technical reasons it is convenient to replace the collection $\{E_n\}$ by $\{E_V\}$ indexed by finite dimensional subspaces V of a countably infinite dimensional real vector space $\mathcal{U}$ called a universe.

Toward equivariant spectra (continued)

The homotopy type of E_V depends only on the dimension of V and there are homeomorphisms

$$E_V \to \Omega^{|W|-|V|} E_W \qquad \text{for } V \subset W \subset \mathcal{U}.$$



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A map of spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.

G-equivariant spectra

Let G be a finite group. Experience has shown that in order to do equivariant stable homotopy theory, one needs G-spaces E_V indexed by finite dimensional orthogonal representations V sitting in a countably infinite dimensional orthogonal representation $\mathcal{U}$.

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This universe $\mathcal{U}$ is said to be complete if it contains infinitely many copies of each irreducible representation of G. A canonical example of a complete universe for finite G is the direct sum of countably many copies of the regular real representation of G.

G-equivariant spectra (continued)

A G-equivariant spectrum (G-spectrum for short) indexed on $\mathcal{U}$ consists of a based G-space E_V for each finite dimensional subspace $V \subset \mathcal{U}$ together with a transitive system of based G-homeomorphisms

$$E_V \xrightarrow[\cong]{\tilde{\sigma}_{V,W}} \Omega^{W-V} E_W$$

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for $V \subset W \subset \mathcal{U}$. Here $\Omega^V X = F(S^V, X)$ and $W - V$ is the orthogonal complement of V in W. As in the classical case, the G-homotopy type of E_V depends only on the isomorphism class of V.

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A suspension G-prespectrum is a G-prespectrum in which the maps above are G-equivalences for V sufficiently large.

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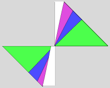

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We define S^{-V} by saying its Wth space for $V \subset W$ is S^{W-V}. This is the analog of formal desuspension in the nonequivariant case.

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$RO(G)$-graded homotopy groups (continued)

Given a virtual representation $\nu = W - V$, we define $S^{\nu} = \Sigma^{W} S^{-V}$. Hence we have a collection of sphere spectra graded over the orthogonal representation ring $RO(G)$.

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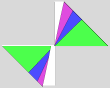
We define

$$\pi_{\nu}^{G}(X) = [S^{\nu}, X]_{G},$$

the $RO(G)$-graded homotopy groups of the G-spectrum X.

MU as a C_2-spectrum

Let ρ denote the real regular representation of C_2.



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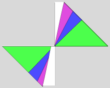
We define a C_2-prespectrum *mu* by $mu_{k\rho} = MU(k)$, the Thom space of the universal $\mathbf{C}^k$-bundle over $BU(k)$, which is a direct limit of complex Grassmannian manifolds.



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Since any orthogonal representation V of C_2 is contained in $k\rho$ for $k \gg 0$, we can define the C_2-spectrum MU by

$$MU_V = \lim_{\substack{\rightarrow \\ k}} \Omega^{k\rho - V} MU(k).$$

MU as a C_2-spectrum (continued)

This spectrum in known as real cobordism theory $MU_{\mathbf{R}}$ and has been studied by Landweber, Araki, Hu-Kriz and Kitchloo-Wilson.

Peter Landweber

Shoro Araki 1930–2005

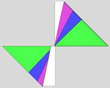
Nitu Kitchloo

Steve Wilson

Igor Kriz and Po Hu

Inducing and coinducing up to a larger group

Let $H \subset G$ be groups and let X be a H-space. There are two ways to get a G-space from it. The corresponding functors are the left and right adjoints to the forgetful functor from G-spaces to H-spaces.



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An example is the the cell-sphere pair

$$(G/H \times D^n, G/H \times S^{n-1}).$$

Inducing and coinducing up to a larger group (continued)

There is also the coinduced G-space

$$\mathrm{map}_H(G, X) \quad = \quad \{f \in \mathrm{map}(G, X) \colon f(\gamma\eta^{-1}) = \eta f(\gamma) \\ \forall \eta \in H \text{ and } \gamma \in G\}$$

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It extends to H-spectra. For a H-spectrum X we denote the coinduced G-spectrum by $N_H^G X$, the norm of X along the inclusion $H \subset G$.

Norming up from MU

We apply this construction to the case $H = C_2$, $G = C_{2^{n+1}}$ and $X = MU_{\mathbf{R}}$.

Mike Hill
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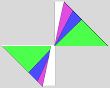

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Let $\gamma \in G$ be a generator and let z_i be a point in $MU_{\mathbf{R}}$. Then the action of G on $MU_{\mathbf{R}}^{(2^n)}$ is given by

$$\gamma(z_1 \wedge \cdots \wedge z_{2^n}) = \overline{z}_{2^n} \wedge z_1 \wedge \cdots \wedge z_{2^n-1},$$

where $\overline{z}_{2^n}$ is the complex conjugate of z_{2^n}.

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(i) It is a C_8-equivariant spectrum, while E_4 was merely an ordinary spectrum with a C_8 "action" for which a homotopy fixed point set could be defined.

(ii) The action of C_8 on $\pi_*(MU_{\mathbf{R}}^{(4)})$ is transparent, unlike its mysterious action on $\pi_*(E_4)$.

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We need to describe the homotopy of the underlying nonequivariant spectrum, which we denote $\pi_*^u(MU_{\mathbf{R}}^{(4)})$.

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$

Recall that $H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$. b_i is the image of a suitable generator of $H_{2i}(\mathbf{C}P^\infty)$ under the map

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The action of γ on these generators is given by

$$\gamma(b_i(j)) = \begin{cases} b_i(j+1) & \text{for } 1 \leq j \leq 3 \\ (-1)^i b_i(1) & \text{for } j = 4. \end{cases}$$

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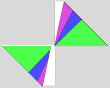

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$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

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Mike Hill
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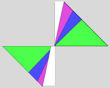

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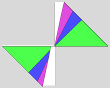

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Suppose X is a G-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension k whose underlying spheres represent a basis of $\pi_k^u(X)$.

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

Recall that in $\pi_*^u(MU_{\mathbf{R}})$, any monomial in the polynomial generators in dimension $2m$ is represented by an equivariant map from $S^{m\rho_2}$.

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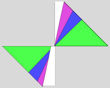

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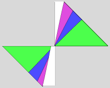

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Recall that the underlying spectrum of $\widehat{W}_1$ is a wedge of 4 copies of S^2.

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

In $\pi_4^u(MU^{(4)})$ there are 14 monomials that fall into 4 orbits under the action of G, each corresponding to a map from a slice cell.

Mike Hill
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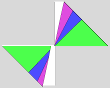
A similar analysis can be made in any even dimension. G always permutes monomials up to sign.



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