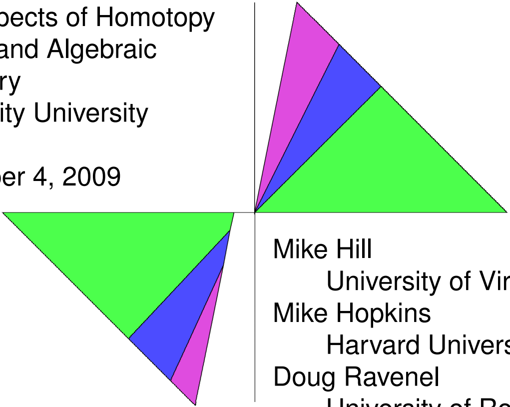
Lecture 2: Formal groups laws and the Hopkins-Miller Theorem

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
Tokyo City University

November 4, 2009



Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester

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Main Theorem

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(iii) $\pi_{-2}(\Omega) = 0$.

The spectrum Ω (continued)

We will construct an equivariant C_8-spectrum $\tilde{\Omega}$ and show that its homotopy fixed point set $\tilde{\Omega}^{hC_*}$ (to be defined below) and its actual fixed point set $\tilde{\Omega}^{C_8}$ are equivalent.

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- The homotopy of $\tilde{\Omega}^{hC_*}$ can be computed using a spectral sequence similar to that of Hopkins-Miller. Twenty year old algebraic methods can be used to show that it detects the θ_js.
- In order to establish (ii) and (iii), we will use equivariant methods to construct a new spectral sequence (the slice spectral sequence) converging to the homotopy of the actual fixed point set $\tilde{\Omega}^{C_8}$.

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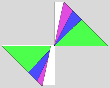

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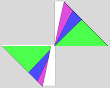
- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.



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- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.
- $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. This is the unoriented cobordism ring.

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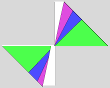

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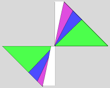
(iii) (Associativity) $F(x,F(y,z)) = F(F(x,y),z)$. This implies more complicated relations among the $a_{i,j}$.

Examples of formal group laws

- $x + y$, the additive formal group law.



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- $x + y + xy$, the multiplicative formal group law. Note here that $1 + F(x, y) = (1 + x)(1 + y)$.
- The power series expansion of $(x + y)/(1 - xy)$. It is associative because

$$\tan(x + y) = F(\tan(x), \tan(y)).$$

Another example of a formal group law

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It is originally due to Euler, see De integratione aequationis differentialis $(m\mathrm{d}x)/\sqrt{1-x^4} = (n\mathrm{d}y)/\sqrt{1-x^4}$, 1753.

The Lazard ring and the universal formal group law

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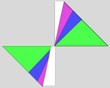
$$L = \mathbf{Z}[a_{i,j}]/(\text{relations})$$

where the relations are those implied by the definition of a formal group law.



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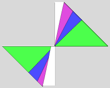

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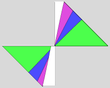

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where $\lambda(G(x, y))$ is the formal group law over R obtained from G by applying λ to each of the $a_{i,j}$.

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The isomorphism is defined (via the uiversal porperty of L) by a formal group law over $\pi_*(MU)$ defined as follows.

Quillen's theorem (continued)

There is a cohomology theory associated with MU under which

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MU^*(\mathbf{C}P^\infty) &= \pi_*(MU)[[x]] \\
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$$MU^*(\mathbf{C}P^\infty) \to MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty)$$

that sends x to a power series in $x \otimes 1$ and $1 \otimes x$ which is a formal group law over $\pi_*(MU)$.

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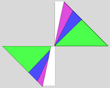
This means that the internal structure of *MU*, and the associated homology and cohomology theories, is intimately related to the structure of formal group laws.

Some relatives of *MU*

Here is an example of this connection.



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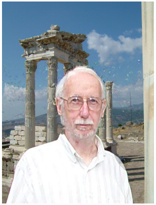
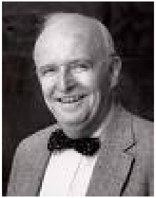
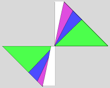




Brown and Peterson originally constructed it (in 1967) via its Postnikov tower.



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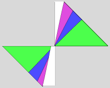

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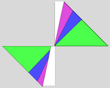
The Brown-Peterson splitting is the topological analog of Cartier's theorem.



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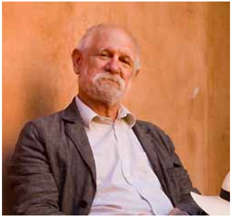

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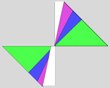

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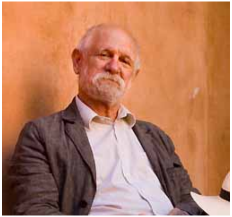

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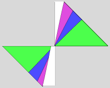
(i) Adjoin a $(p^n - 1)$the root of v_n, invert it and kill the higher generators.



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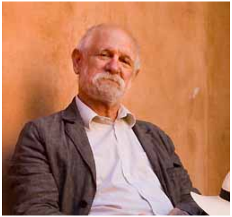

The Morava spectrum E_n (for a positive integer n) is an E_∞-ring spectrum such that $\pi_*(E_n)$ obtained from $\pi_*(BP)$ as follows:

(i) Adjoin a $(p^n - 1)$the root of v_n, invert it and kill the higher generators.

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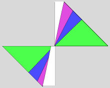

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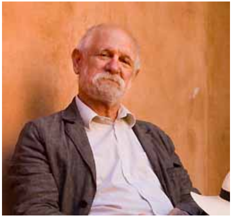

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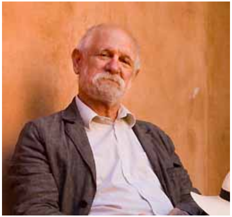

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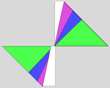
The ring $\pi_*(E_n)$ was studied by Lubin-Tate. They showed that it classifies liftings (to Artinian rings) of a certain formal group law F_n over $\mathbf{F}_{p^n}$, the Honda formal group law.



The Morava stabilizer group S_n

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Mike Hill
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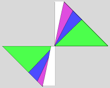
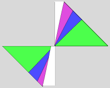
S_n is the automorphism group of the Honda formal group law F_n. It a crucial ingredient in chromatic stable homotopy theory.

Its action on F_n lifts to an action on $\pi_*(E_n)$, the Lubin-Tate ring. This action is defined by certain formulas but is mysterious in practice.



The Morava stabilizer group S_n (continued)

S_n is a finite extension of a pro-p-group isomorphic to a group of units in a certain division algebra D_n of rank n^2 over the p-adic numbers $\mathbf{Q}_p$.



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We will be interested in some finite subgroups of S_n.

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The Hopkins-Miller theorem

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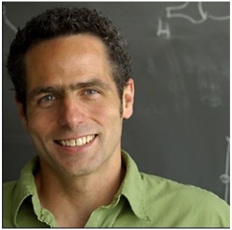

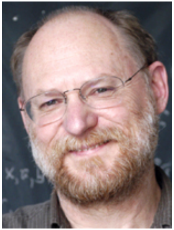

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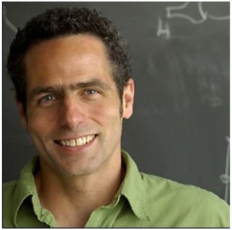

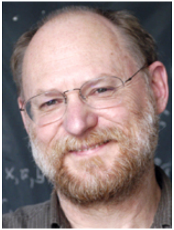

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$E_n^{hS_n}$ is $L_{K(n)}S^0$, the localization of the sphere spectrum with respect to the nth Morava K-theory.

The Hopkins-Miller theorem (continued)

Hopkins-Miller Theorem (1992?)

For each closed subgroup $G \subset S_n$ there is a homotopy fixed point set E_n^{hG} and a spectral sequence

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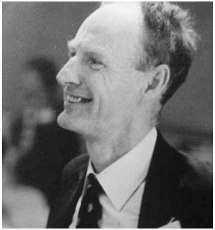

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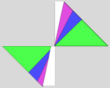
- $C_8 \subset S_4$. The division algebra D_4 contains eighth roots of unity for similar reasons.



A first attempt to define the magic spectrum Ω

- The spectrum $E_4^{hC_8}$ can be shown to satisfy the first condition required of Ω, namely its Adams-Novikov spectral sequence detects all of the θ_js.

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- The one for $E_4^{hC_8}$ is too complicated for us to use it to prove that $\pi_{-2} = 0$.

A C_8-equivariant substitute for E_4

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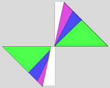

The Hopkins-Miller spectral sequence for *KO*

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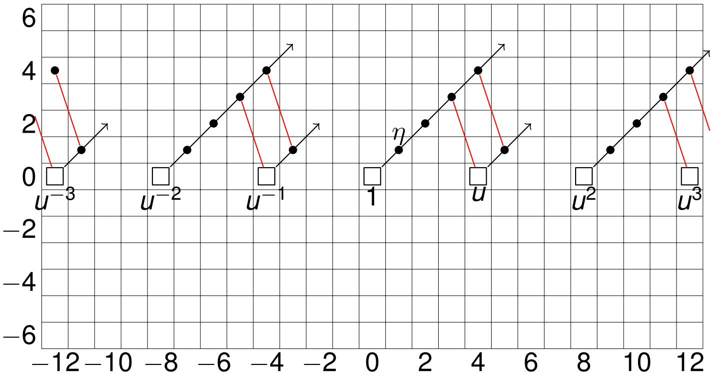
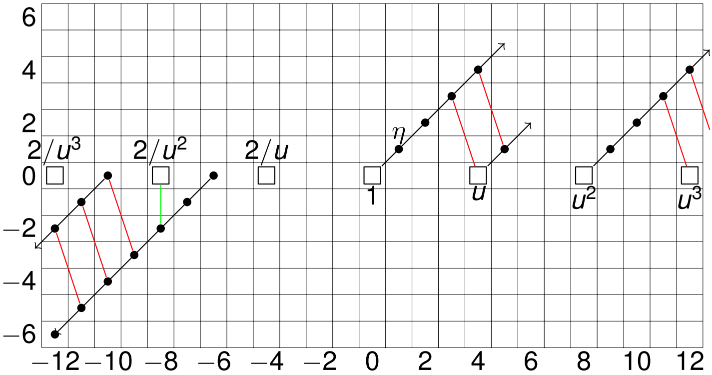
It has a slice spectral sequence that was the subject of Dan Dugger's thesis.

The Hopkins-Miller spectral sequence for *KO* (continued)



Here is the Hopkins-Miller spectral sequence it.

The slice spectral sequence for *KO*



Here is the slice spectral sequence for the actual fixed point set.

Actual fixed points and homotopy fixed points

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With this in mind, comparing the two E_2-terms enables us to determine the complete behavior of each SS.