Lecture 1: History and background of the problem

A solution to the
Arf-Kervaire invariant
problem

New Aspects of Homotopy
Theory and Algebraic
Geometry
Tokyo City University

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The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.

A wildly popular dance craze

Our main result (continued)

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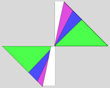
The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2+n}(S^n)$ for large n do not exist for $j \geq 7$.



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The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2.

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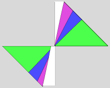

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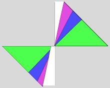

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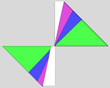

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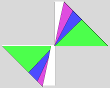

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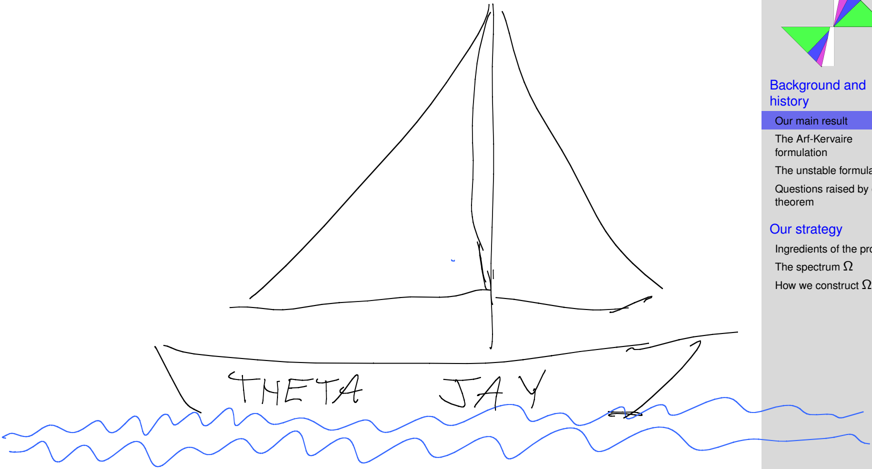

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In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.

On the money: Arf's definition republished in 2009

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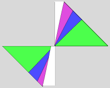

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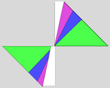

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The *Kervaire invariant* $\Phi(M)$ is defined to be the Arf invariant of q.

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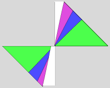

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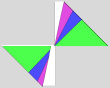

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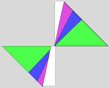

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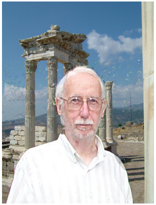
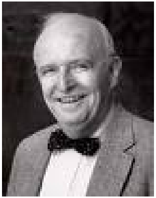
-   Brown-Peterson (1966) showed that it vanishes for all positive even k.

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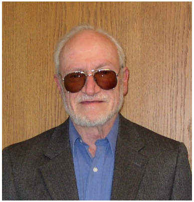
- 
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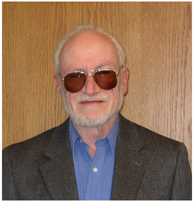
- 

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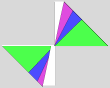

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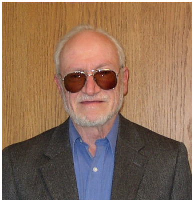
- 

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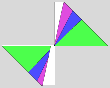

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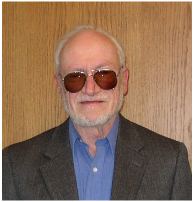
- 

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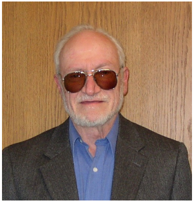
- 

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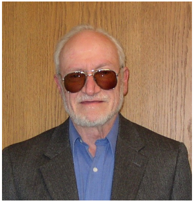
- 

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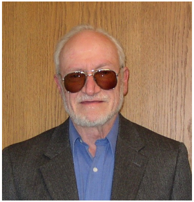
- 

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This leads to a long exact sequence of homotopy groups

$$\cdots \to \pi_m(S^n) \xrightarrow{E} \pi_{m+1}(S^{n+1}) \xrightarrow{H} \pi_{m+1}(S^{2n+1}) \xrightarrow{P} \pi_{m-1}(S^n) \to \cdots$$

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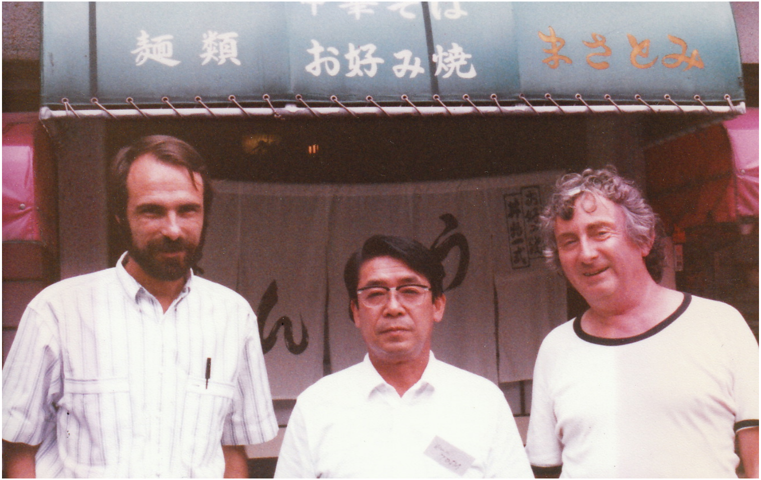

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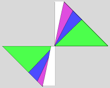

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- For such n, w_n is divisible by 2 iff $n = 2^{j+1} - 1$ with $j > 2$ and θ_j exists, in which case $w_n = 2\theta_j$.

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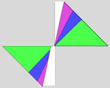

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k	1	2	3	4	5	6	7	8	9	10
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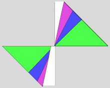

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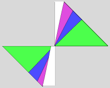

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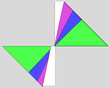

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Each Whitehead square $w_{2n+1} \in \pi_{4n+1}(S^{2n+1})$ (except the cases $n = 0$, 1 and 3) desuspends to a lower sphere until we get an element with a nontrivial Hopf invariant, which is always some β_j.

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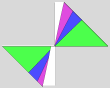

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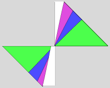
for each $j > 0$ and $s \geq 0$. This result is essentially Adams' 1961 solution to the vector field problem.



Back to the EHP sequence

Recall the EHP sequence

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- *Let $j, s > 0$ and suppose that $m = 2^{j+2}(s+1) - 4 - \phi(j)$ and $n = 2^{j+1}(s+1) - 2 - \phi(j)$. Then $P(\beta_j)$ has Hopf invariant θ_j.*

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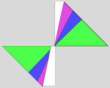

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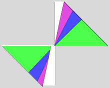
Ingredients of the proof

Our proof has several ingredients.



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Our proof has several ingredients.

- It uses methods of stable homotopy theory, which means it uses spectra instead of topological spaces.

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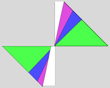

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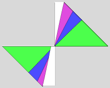

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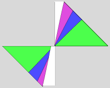

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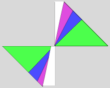

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If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

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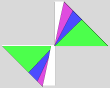

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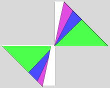

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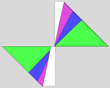

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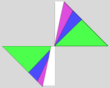
To get a C_8-spectrum, we use the following general construction for getting from a space or spectrum X acted on by a group H to one acted on by a larger group G containing H as a subgroup.



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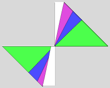

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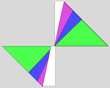

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This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.