Topology Vol. 15, pp. 149–153. Pergamon Press, 1976. Printed in Great Britain

THE STRUCTURE OF BP_*BP MODULO AN INVARIANT PRIME IDEAL

DOUGLAS C. RAVENEL†

(*Received* 16 *October* 1975)

THE PURPOSE of this note is to show how the structure formulae for BP_*BP originally given by Quillen in [5] and subsequently by Adams in [1], can be simplified when one passes to

$$BP_*BP/I_n = BP_*/I_n \otimes_{BP_*} BP_*BP = BP_*BP \otimes_{BP_*} BP_*/I_n$$

where $I_n = (p, v_1, \ldots, v_{n-1}) \subset BP_*$ is the nth invariant prime ideal (see [3]). The generators v_i will be defined below. We will obtain simplification of the formulae for the right unit of v_{i+n} (Theorems 6 and 7) and the coproduct (Theorem 8) and conjugation (Theorem 9) of t_i, for $i \le 2n$. We will also obtain an expression for the formal group law over BP_*/I_n (Theorem 5).

These results will be applied to problems related to the Adams–Novikov spectral sequence in a series of papers beginning with [4].

I wish to thank Steve Wilson and Dave Johnson for suggesting some improvements in the manuscript, and Manuel Moreira for pointing out some errors in an earlier version of this paper. I am also grateful to the Institute for Advanced Study for providing financial support and pleasant surroundings while this paper was being written.

We will use the notation of [1].

Recall $BP_* = Z_{(p)}[v_1, v_2, \ldots]$ where $\dim v_i = 2(p^i - 1)$, $BP_* \otimes \mathbf{Q} = \mathbf{Q}[l_1, l_2, \ldots]$ where $l_i = m_{p^i-1} = (CP^{p^i-1}/p^i)$ and $BP_*BP = BP_*[t_1, t_2, \ldots]$ where $\dim t_i = 2(p^i - 1)$. The structure maps are $\eta_R : BP_* \to BP_*BP$ given by

$$\eta_R(l_n) = \sum_{0 \le i \le n} l_i t_{n-i}^{p^i}, \quad (t_0 = 1) \tag{1}$$

the coproduct $\psi : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$ given by

$$\sum_{i+j=k} l_i (\psi t_j)^{p^i} = \sum_{h+i+j=k} l_h t_i^{p^h} \otimes t_j^{p^{h+i}}, \tag{2}$$

and the conjugation $c : BP_*BP \to BP_*BP$ given by

$$\sum_{h+i+j=k} l_h t_i^{p^h} c(t_j)^{p^{h+i}} = l_k. \tag{3}$$

We will also use the formula of Hazewinkel ([2]), i.e. that multiplicative generators $v_i \in BP_*$ can be defined by the inductive formula

$$v_n + \sum_{0<i<n} l_i v_{n-i}^{p^i} = p l_n. \tag{4}$$

There is a formal group law $F(x, y) \in BP_*[[x, y]]$ associated with BP given by

$$F(x, y) = \exp(\log x + \log y) \tag{5}$$

where

$$\log x = \sum_{i \ge 0} l_i x^{p^i} \qquad (l_0 = 1) \tag{6}$$

and

$$\exp(\log x) = x. \tag{7}$$

We will denote

$$\exp\left(\sum_i \log a_i\right) \quad \text{by} \quad \sum_i^F a_i.$$

†Partially supported by NSF grant MPS72-05055 A02.

Notice that $F(x, y) \in BP_*[[x, y]]$ even though $\exp x, \log x \in (Q \otimes BP_*)[[x]]$.

Our first result is an expression for the right unit of the Hazewinkel generators mod p. This will be further simplified below.

THEOREM 1.

$$\sum_{\substack{i\geq 0\\ j>0}}^{F} t_i\eta_R(v_j)^{p^i} \equiv \sum_{\substack{i>0\\ j\geq 0}}^{F} v_i t_j^{p^i} \bmod (p). \tag{8}$$

Proof. We apply η_R to (4) and get

$$\sum_{0\leq i<n} \eta_R(l_i)\eta_R(v_{n-i})^{p^i} = p\eta_R(l_n) \quad \text{for } n>0.$$

Then apply (1) to both sides and get

$$\sum_{0\leq i+j<n} l_i t_j^{p^i}\eta_R(v_{n-i-j})^{p^{i+j}} = p\sum_{0\leq i\leq n} l_i t_{n-i}^{p^i}.$$

Then substitute (4) in the right-hand side and get

$$\sum_{0\leq i+j<n} l_i t_j^{p^i}\eta_R(v_{n-i-j})^{p^{i+j}} = pt_n + \sum_{0\leq j<i\leq n} l_j v_{i-j}^{p^j} t_{n-i}^{p^i}.$$

Summing over all $n>0$ gives

$$\sum_{\substack{i\geq 0\\ j>0}} \log t_i\eta_R(v_j)^{p^i} = \sum_{n>0} pt_n + \sum_{\substack{i>0\\ j\geq 0}} \log v_i t_j^{p^i}.$$

Applying exp gives us

$$\sum_{\substack{i\geq 0\\ j>0}}^{F} t_i\eta_R(v_j)^{p^i} = F\left(\exp\sum_{n>0} pt_n, \sum_{\substack{i>0\\ j\geq 0}}^{F} v_i t_j^{p^i}\right).$$

Hence it suffices to show that each term of $\exp \sum_{n>0} pt_n$ is in pBP_*BP, which is a consequence of the following:

LEMMA 2. *The element* $\exp px \in (\mathbf{Q} \otimes BP_*)[[x]]$ lies in $pBP_*[[x]]$.

Proof. First note that for any $t \in BP_*[[x]]$, $\log pt \in pBP_*[[x]]$ since $p^i l_i \in BP_*$. Now let $\exp px = \sum_{i>0} u_i x^i$ with $u_i \in \mathbf{Q} \otimes BP_*$. We will show inductively that $u_i \in pBP_*$. Now $\log(\exp px) = px$ so $u_1 = p$. Suppose $u_i \in pBP_*$ for $i<k$. Then $px = \log \sum_{i=0} u_i x^i \equiv u_k x^k + y_k \bmod (x^{k+1})$ where $y_k \in pBP_*[[x]]$. Hence $u_k \in pBP_*$.

□

In order to simplify (2), (3) and (8) modulo I_n we will need to get a more explicit expression for the formal group law. First we need the following result about pth powers.

LEMMA 3. *Let A and B be ideals in a commutative $Z_{(p)}$ algebra with $A \subset B$, and let $x, y \in B$ such that $x \equiv y$ mod A. Then*

$$x^{p^n} \equiv y^{p^n} \bmod \sum_{0\leq i\leq n} (p^{n-i})B^{p^n-p^i}A^{p^i}.$$

Proof. Let $x = y + a$ with $a \in A$. Then

$$\begin{aligned} x^{p^n} - y^{p^n} &= \sum_{j>0} \binom{p^n}{j} y^{p^n-j}a^j \\ &\in \sum_{j>0} \binom{p^n}{j} B^{p^n-j}A^j \\ &\supset \sum_{0\leq i\leq n} (p^{n-i})B^{p^n-p^i}A^{p^i}. \end{aligned}$$

To show the opposite inclusion we consider the terms in which j is not a power of p. Let $j = cp^i$ with $c > 1$ and $(c, p) = 1$. Then

$$\binom{p^n}{j} B^{p^n - cp^i} A^{cp^i} = (p^{n-i}) B^{p^n - cp^i} A^{(c-1)p^i} A^{p^i}$$
$$\subset (p^{n-i}) B^{p^n - cp^i} B^{(c-1)p^i} A^{p^i}$$
$$= (p^{n-i}) B^{p^n - p^i} A^{p^i}. \qquad \square$$

We now define a symmetric polynomial in k-variables for $m > 0$

$$C_{p^m}(x_1, \ldots, x_k) = \frac{(\Sigma x_i)^{p^m} - \Sigma(x_i^{p^m})}{p}$$

with the following elementary properties. (We will often abbreviate the above by $C_{p^m}(x_i)$).

PROPOSITION 4.
(a) $C_{p^m}(x_1, \ldots, x_k)$ *has integer coefficients*
(b) $C_{p^m}(x_i) \equiv C_p(x_i)^{p^{m-1}} \equiv C_p(x_i^{p^{m-1}}) \bmod p$
(c) $C_{p^m}(x_i + x_2, x_3, \ldots) = C_{p^m}(x_1, x_2, x_3, \ldots) - C_{p^m}(x_1, x_2)$. $\square$

Now let $J_m \subset R[[x_1, \ldots, x_k]]$ for a commutative ring R denote the ideal generated by monomials of degree $\geq m$. Then we have

THEOREM 5. *The formal group law over BP_*/I_n is given by*

$$\sum_{i=1}^{k} {}^F x_i \equiv \left(\sum_i x_j\right) - \sum_{0 \leq i < n} v_{n+i} C_{p^{n+i}}(x_1, \ldots, x_k)$$
$$\text{modulo } J_{p^{2n-1} + (p-1)p^{n-1}}$$

Proof. Let $z = F(x_1, \ldots, x_k)$. If we reduce Hazewinkel's formula (4) to $BP_*/(v_1, \ldots, v_{n-1}) \otimes \mathbf{Q}$ we get

$$\log x \equiv x + \sum_{0 \leq i < n} v_{n+i} \frac{x^{p^{n+i}}}{p} \bmod J_{p^{2n}}. \tag{5.1}$$

In particular

$$\log x \equiv x \bmod J_{p^n}$$

so

$$z \equiv \Sigma x_i \bmod J_{p^n}. \tag{5.2}$$

Since $\log z = \Sigma \log x_i$, by 5.1 we have

$$z + \sum_{0 \leq j < n} v_{n+i} \frac{z^{p^{n+j}}}{p} \equiv \sum_i x_i + \sum_{\substack{0 \leq j < n \\ i}} v_{n+i} \frac{x_i^{p^{n+j}}}{p} \bmod J_{2p^n},$$

or

$$z \equiv \sum_i x_i + \sum_{0 \leq j < n} \frac{v_{n+j}}{p} \left(\sum_i x_i^{p^{n+j}} - z^{p^{n+j}} \right) \bmod J_{p^{2n}}. \tag{5.3}$$

Now if we apply Lemma 3 (setting $A = J_{p^n}$ and $B = J_1$) to 5.2 we see that

$$z^{p^{n+j}} \equiv (\Sigma x_i)^{p^{n+j}} \bmod (p^2) + (p) J_{(p-1)k^{j+n-1} + p^{2n+j-1}} + J_{p^{2n+j}}. \tag{5.4}$$

If we substitute 5.4 in the right-hand side of 5.3, we get the desired expression for z. $\square$

We are now in a position to make (2), (3) and (8) more explicit.

THEOREM 6. *In BP_*BP/I_n for $k \leq n$*

$$\sum_{0 \leq i \leq k} t_i \eta_R(v_{n+k-i})^{p^i} = \sum_{0 \leq i \leq k} v_{n+i} t_{k-i}^{p^{n+i}} \tag{9}$$

i.e.,

$$\eta_R(v_{n+k}) = \sum (-1)^m t_{i_1} t_{i_2}^{a_1} \ldots t_{i_m}^{a_{m-1}} v_{n+j_1}^{a_m} t_{j_2}^{d} \tag{10}$$

where the sum is over all subscripts with $i_1 + i_2 + \cdots + i_m + j_1 + j_2 = k$, $0 \leq m \leq$, $i_s > 0$, $j_s \geq 0$, and *where* $a_1 = p^{i_1}$, $a_s = p^{i_s}a_{s-1}$ *and* $d = a_m p^{n+j_1}$.

Proof. Modulo I_n (8) becomes

$$\sum_{\substack{i\geq 0\\ j\geq n}}^{F} t_i\eta_R(v_j)^{p^i} \equiv \sum_{\substack{i\geq n\\ j\geq 0}}^{F} v_i t_j^{p^i}.$$

Since $\eta_R(v_n) = v_n$, we can rewrite this as

$$F\Big(v_n, {}'\sum_{i,j\geq 0}^{F} t_i\eta_R(v_{n+j})^{p^i}\Big) \equiv F\Big(v_n, {}'\sum_{i,j\geq 0}^{F} v_{n+i}t_j^{p^{n+i}}\Big)$$

i.e.

$${}'\sum_{i,j\geq 0}^{F} t_i\eta_R(v_{n+j})^{p^i} \equiv {}'\sum_{i,j\geq 0}^{F} v_{n+i}t_j^{p^{n+i}}$$

where the symbol $'\Sigma^F$ indicates that the term with $i = j = 0$ is excluded. It follows from Theorem 5 that the expression for $\eta_R(v_{n+k})$ will not involve any of the cross terms in the formal group law if $k \leq n$, so we get the first formula of the Theorem. The derivation of the second formula from the first is routine.

□

Examples. In BP_*BP/I_n we have

$$\eta_R(v_{n+1}) = v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \quad \text{for } n \equiv 1 \tag{11}$$

$$\begin{aligned}\eta_R(v_{n+2}) = {}& v_{n+2} + v_{n+1}t_1^{p^{n+1}} + v_n t_2^{p^n} - v_{n+1}^p t_1 - v_n^{p^2}t_2\\ &+ t_1^{1+p}v_n^{p^2} - t_1^{1+p^{n+1}}v_n^p \quad \text{for } n \geq 2\end{aligned} \tag{12}$$

$$\begin{aligned}\eta_R(v_{n+3}) = {}& v_{n+3} + v_{n+2}t_1^{p^{n+2}} + v_{n+1}t_2^{p^{n+1}} + v_n t_3^{p^n} - v_{n+2}^p t_1\\ &- v_{n+1}^{p^2}t_2 - v_n^{p^3}t_3 - t_1^{1+p^{n+2}}v_{n+1}^p - t_1 t_2^{p^{n+1}}v_n^p - t_2 t_1^{p^{n+2}}v_n^{p^2}\\ &+ t_1^{1+p}v_{n+1}^{p^2} + t_1 t_2^p v_n^{p^3} + t_2 t_1^{p^2}v_n^{p^3} + t_1^{1+p+p^{n+2}}v_n^{p^2} - t_1^{1+p+p^2}v_n^{p^3}\\ &\text{for } n \geq 3.\end{aligned} \tag{13}$$

THEOREM 7. *In* BP_*BP/I_n *for* $n < k \leq 2n$

$$\begin{aligned}&\sum_{0\leq i\leq k} t_i\eta_R(v_{n+k-i})^{p^i} - \sum_{0\leq j\leq k-n-1} v_{n+j}C_{p^{n+j}}(\eta_R(v_{k-j}), t_1\eta_R(v_{k-j-1})^p, \ldots, t_{k-j-n}v_n^{p^{k-j-n}})\\ &\qquad = \sum_{0\leq i\leq k} v_{n+i}t_{k-i}^{p^{n+i}} - \sum_{0\leq j\leq k-n-1} v_{n+j}C_{p^{n+j}}(v_{k-j}, v_{k-j-1}t_1^{p^{k-j-1}}, \ldots, v_n t_{k-j-n}^{p^n}).\end{aligned} \tag{14}$$

Proof. In the mod I_n reduction of (8) the first terms (after cancelling v_n on both sides) occur in dimension $2(p^{n+1}-1)$. Hence Theorem 5 is sufficient for getting an explicit expression up to dimension $2(p^{n+1}-1)(p^{2n-1}+(p-1)p^{n-1})$, which is greater than $2(p^{3n}-1)$ and therefore adequate for our purposes. The right-hand side of (8) becomes (in our range)

$$\sum_{\substack{i>0\\ j\geq 0}} v_{n+i}t_j^{p^{n+i}} - \sum_{0\leq m<n} v_{n+m}C_p(v_{n+l}^{p^{n+m-1}}t_h^{p^{2n+l+m-1}})$$

and the left-hand expression has a similar form. The variables to which $C_p(\ )$ is applied all have dimensions of the form $2(p^a - p^b)$ with $b \geq n$ and $a \geq 2n$, so it follows that the only terms that will have the dimension of v_{n+k} are those indicated above.

□

Examples. For $n = 1$, $p > 2$

$$\begin{aligned}\eta_R(v_3) = {}& v_3 + v_2 t_1^{p^2} + v_1 t_2^p - v_2^p t_1 - v_1^{p^2}t_2 - v_1^p t_1^{1+p^2} + v_1^{p^2}t_1^{1+p}\\ &- v_1 C_p(v_2, v_1 t_1^p, -v_1^p t_1),\end{aligned} \tag{15}$$

and for $n = 2$, $p > 2$

$$\begin{aligned}\eta_R(v_5) = {}& v_5 + v_4 t_1^{p^4} + v_3 t_2^{p^3} + v_1 t_3^{p^2} - v_4^p t_1 - v_3^{p^2}t_2 - v_2^{p^3}t_3 - v_3^p t_1^{1+p^4}\\ &- v_2^p t_1 t_2^{p^3} - v_2^{p^2}t_1^{p^4}t_2 + v_3^{p^2}t_1^{1+p} + v_2^{p^3}t_1 t_2^p + v_2^{p^3}t_1^{p^2}t_2 + v_2^{p^2}t_1^{1+p+p^4}\\ &- v_2^{p^3}t_1^{1+p+p^2} - v_2 C_{p^2}(v_3, v_2 t_1^{p^2}, -v_2^p t_1).\end{aligned} \tag{16}$$

For $p = 2$, add $v_1^5 v_1^2$ to (15) and $v_2^9 t_1^4$ to (16).

We now turn our attention to the coproduct. (2) can be rewritten as

$$\sum_{i\geq 0}^{F} \psi(t_i) = \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i} \tag{17}$$

so we get

THEOREM 8. *In BP_*BP/I_n for $k \leq 2n$*

$$\psi(t_k) = \sum_{0\leq i\leq k} t_i \otimes t_{k-i}^{p^i} - \sum_{0\leq j\leq k-n-1} v_{n+j}C_{p^{n+j}}(t_i \otimes t_{k-n-j-i}^{p^i}). \tag{18}$$

Proof. The argument is similar to that of Theorem 7 and is left to the reader. □

For the conjugation we can rewrite (3) as

$$\sum_{i,j\geq 0}^{F} t_i c(t_j)^{p^i} = 1. \tag{19}$$

Let $x_k \in BP_*BP$ be defined inductively by $x_o = 1$ and

$$\sum_{0\leq i\leq k} t_i x_{k-i}^{p^i} = 0 \quad \text{for } k > 0.$$

Explicitly we have

$$x_k = \sum (-1)^m t_{i_1} t_{i_2}^{a_1} \ldots t_{i_m}^{a_{m-1}} \tag{20}$$

where the notation is similar to that of Theorem 6. Then we have

THEOREM 9. *In BP_*BP/I_n for $k \leq 2n$*

$$c(t_k) = x_k + \sum_{0\leq j<k-n} v_{n+j}C_{p^{n+j}}(t_h x_{k-n-j-h}^{p^h})$$
$$- \sum_{\substack{0<i<k-n \\ 0\leq j<k-n-i}} t_i v_{n+j}^{p^i} C_{p^{n+i+j}}(t_h x_{k-n-j-h}^{p^h}). \tag{21}$$

Proof. This follows from (19) by the methods used in the proof of Theorem 7. □

REFERENCES

1. J. F. ADAMS: *Stable homotopy and generalized homology*, University of Chicago, 1974.
2. M. HAZEWINKEL: Constructing formal groups I. Over $\mathbf{Z}_{(p)}$-algebras, *Report* 7119, *Netherlands School of Economics, Econometric Institute.*
3. P. LANDWEBER: Annihilator ideals and primitive elements in complex cobordism, *Illinois J. Math.* 17 (1973), 273–284.
4. H. R. MILLER and D. C. RAVENEL: Morava stabilizer algebras and the localization of Novikov's E_2-term (to appear).
5. D. QUILLEN: On the formal group laws of unoriented and complex cobordism theory, *AMS Bulletin* 75 (1969), 1293–1298.

Institute for Advanced Study
Princeton, New Jersey 08540
Columbia University
New York, N.Y. 10027