Some early and middle mathematical work of Bob Stong

Doug Ravenel

University of Rochester

November 10, 2007

Bob's early career

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- 1959-62: M.S. and Ph.D. at University of Chicago under Dick Lashof
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The mathematical landscape in 1962

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- The Adams-Novikov spectral sequence, 1967
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Mod 2 cohomology of the connective covers of BO and BU

"Determination of $H^*(\mathrm{BO}(k, \cdots, \infty), Z_2)$ and $H^*(\mathrm{BU}(k, \cdots, \infty), Z_2)$. " appeared in the AMS Transactions in 1963.

It was a tour de force computation with the Serre spectral sequence and the Steenrod algebra.

It was the first determination of the cohomology of an infinite delooping of an infinite loop space, other than the Eilenberg-Mac Lane spectrum.

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The first three connective covers of BO are

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\begin{aligned}
BO\,(2,\,\infty) &= BSO \\
BO\,(3,\,\infty) = BO\,(4,\,\infty) &= BSpin \\
BO\,(5,\,\infty) = \cdots = BO\,(8,\,\infty) &= BString
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The map from $H^*(BO)$ to the cohomology of each of these is onto.

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The integral cohomology of each of them is torsion free, which is not true of the higher connective covers.

The Thom spectra $MString$ and $MU(6)$ both figure in the theory of topological modular forms.

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The main result of the paper implies that

$$H^*(bo; \mathbf{Z}/2) = A \otimes_{A(1)} \mathbf{Z}/2$$

where $A(1)$ denotes the subalgebra of the mod 2 Steenrod algebra A generated by Sq^1 and Sq^2,

and

$$H^*(bu; \mathbf{Z}/2) = A \otimes_{Q(1)} \mathbf{Z}/2$$

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Mod 2 cohomology of the connective covers of BO and BU, 1963, methods used

It was known that

$$H^*(BO; \mathbf{Z}/2) = \mathbf{Z}/2[w_1, w_2, \ldots]$$

where $w_i \in H^i$ is the ith Stiefel-Whitney class.

Bob replaced the generator w_i with $\theta_i \in H^i$, which he defined as the image of a certain Steenrod operation acting on w_{2^m}, where $m = \alpha(i-1)$, with $\alpha(j)$ being the number of ones in the dyadic expansion of j.

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For example we have $\theta_1 = w_1$,

$$\theta_2 = w_2 \xrightarrow{Sq^1} \theta_3 \xrightarrow{Sq^2} \theta_5 \xrightarrow{Sq^4} \theta_9 \xrightarrow{Sq^8} \cdots$$

and

$$\begin{tikzcd}
\theta_4 = w_4 \arrow[r, "Sq^3"] \arrow[dr, "Sq^2"'] & \theta_7 \arrow[r, "Sq^6"] & \theta_{13} \arrow[r, "Sq^{12}"] & \theta_{25} \arrow[r, "Sq^{24}"] & \cdots \\
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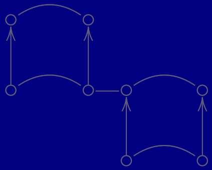
He also makes use of some exact sequences of A-modules discovered by Toda. They can be obtained by applying the functor $A \otimes_{A(1)} (\cdot)$ to the following sequences exact of $A(1)$-modules.

For the BU computation,

$$\begin{array}{c} A/A(Sq^3, Sq^1) \\ \uparrow \\ A/ASq^1 \\ \uparrow \\ \Sigma^3 A/A(Sq^3, Sq^1) \end{array}$$



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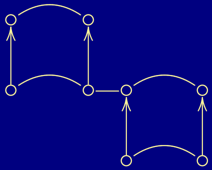

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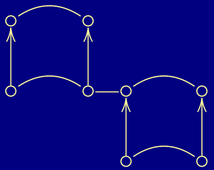

This short exact sequence defines (via a change-of-rings isomorphism relating $A(1)$ and $Q(1)$) an element in

$$\mathrm{Ext}^{1,3}_{Q(1)}(\mathbf{Z}/2, \mathbf{Z}/2)$$

corresponding to the complex form of Bott periodicity.

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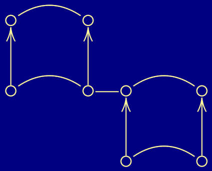

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Mod 2 cohomology of the connective covers of BO and BU, 1963, methods used

The following 6-term exact sequence defines an element in

$$\mathrm{Ext}^{4,12}_{A(1)}(\mathbf{Z}/2, \mathbf{Z}/2)$$

corresponding to the real form of Bott periodicity.

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$$
\begin{array}{c}
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\uparrow \\
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\uparrow \\
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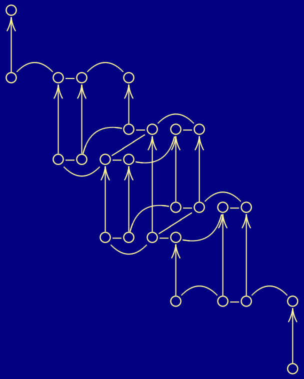

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Within this diagram are four modules of interest:

$A/A(Sq^1, Sq^2)$  A/ASq^2 

A/ASq^3  $A/A(Sq^1, Sq^5)$ 

which we denote by M_0, M_1, M_2 and M_4.

$M_s = H^*(bo\langle s\rangle)$, the mod 2 cohomology of the stable connective cover in the diagram

$$\begin{tikzcd}
bo = bo\langle 0\rangle \arrow[d] & bo\langle 1\rangle \arrow[l] \arrow[d] & bo\langle 2\rangle \arrow[l] \arrow[d] & bo\langle 4\rangle \arrow[l] \\
K(\mathbf{Z}, 0) & K(\mathbf{Z}/2, 1) & K(\mathbf{Z}/2, 2) &
\end{tikzcd}$$

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Mod 2 cohomology of the connective covers of BO and BU, statement of main result

Theorem

Let $s = 0, 1, 2$ or 4, $8k + s > 0$,
and $K_{8k+s} = K(\pi_{8k+s}(BO), 8k + s)$.
Then

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The image of $H^*(K_{8k+s})$ is an unstable version of M_s.

Mod 2 cohomology of the connective covers of BO and BU, statement of main result

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Let $s = 0, 1, 2$ or 4, $8k + s > 0$,
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The Stong-Hattori theorem

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What the theorem says in the complex case

A complex manifold M of dimension n (meaning real dimension $2n$) comes equipped with a map $f : M \to BU$ classifying its stable normal bundle.

This gives us normal Chern classes $f^*(c_i) \in H^{2i}(M)$ for $1 \leq i \leq n$.

Any degree n monomial c^J in these can be evaluated on the fundamental homology class of M, and we get a Chern number

$$\langle c^J, [M] \rangle \in \mathbf{Z}.$$

Chern numbers were shown by Milnor to be complete cobordism invariants, i.e., M is a boundary iff all of its Chern numbers vanish.

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Equivalently, the classifying map f of the normal bundle on M gives us an element $f_*([M]) \in H_{2n}(BU)$ which depends only on the cobordism class of M.

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$$\text{complex cobordism ring} = \pi_*(MU) \to H_*(BU).$$

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What is the image of $\eta : \pi_*(MU) \to H_*(MU)$?

Two facts were already known about this:

(i) Tensoring both side with the rationals converts η into an isomorphism, so the image of η has locally finite index.

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A reformulation of the Stong-Hattori theorem

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Two maps $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$ (where M_1 and M_2 are closed m-dimensional manifolds, while N_1 and N_2 are closed n-dimensional manifolds) are *cobordant* if there is a diagram

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Assume that all manifolds in sight have compatible G-structures in their stable normal bundles for $G = O$, SO, U, etc.

Let $\Omega^G(m, n)$ denote the group of cobordism classes of maps as defined above, and let MG denote the Thom spectrum associated with G.

Bob showed that

$$\begin{aligned} \Omega^G(m, n) &\cong \lim_{r \to \infty} MG_n\left(\Omega^{r+m} MG_{r+n}\right) \\ &\cong MG_n\left(\Omega^{\infty} MG_{n-m}\right) \end{aligned}$$

The cobordism groups of maps are the bordism groups of the spaces in the Ω-spectrum for MG.

He described these groups explicitly for $G = O$.

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Fast forward 20 years: Ochanine's theorem on elliptic genera, 1987

Theorem

Suppose we have a homomorphism $\varphi : \Omega_*^{\mathrm{so}} \to \Lambda$, from the oriented bordism ring to a commutative unital Q-algebra. Then it vanishes on all manifolds of the form $\mathbf{C}\mathrm{P}(\xi)$ with ξ an even-dimensional complex vector bundle over a closed oriented manifold if and only if the logarithm

$$g(u) = \sum_{n \geq 0} \frac{\varphi(\mathbf{C}\mathrm{P}^{2n})}{2n+1} u^{2n+1}$$

of the formal group law of φ is given by an elliptic integral of the first kind, i.e., by

$$g(u) = \int_0^u \frac{dz}{\sqrt{P(z)}}, P(z) = 1 - 2\delta z^2 + \varepsilon z^4, \delta, \varepsilon \in \Lambda.$$

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Ochanine's theorem on elliptic genera, continued

Such a homomorphism $\varphi\, \Omega_*^{\mathrm{so}} \to \Lambda$ is now called an elliptic genus.

If $\delta = \varepsilon = 1$, then φ is the signature.

If $\varepsilon = 0$ and $\delta = -\frac{1}{8}$, then φ is the $\hat{A}$-genus.

We now know that Λ need only be an algebra over the ring $\mathbf{Z}[1/2][\delta, \varepsilon]$, which can be interpreted as a ring of modular forms.

The manifolds on which φ vanishes admit semi-free S^1-actions (since $\mathbf{C}\mathrm{P}^{2n-1}$ does), and $\varphi(M)$ is an obstruction to the existence of such an action.

In the past 20 years there has been a lot of interest in interpreting such a genus geometrically or analytically.

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The 1988 paper of Landweber and Stong

Circle actions on Spin manifolds and characteristic numbers appeared in TOPOLOGY in 1988.

Bob and Peter gave a formula for an elliptic genus with values in $\mathbf{Q}[[q]]$ in terms of KO-characteristic classes,

$$\varphi(M) = \sum_{k=0}^{\infty} \langle \hat{A}(M)\,\mathrm{ch}(\rho_k(TM)),\, [M]\rangle q^k$$

where $\hat{A}(M)$ the total $\hat{A}$-class of the tangent bundle of M, $\mathrm{ch}(E)$ is the Chern character of the complexification of E and $\rho_k(TM)$ is a certain element in $KO^*(M) \otimes \mathbf{Q}$, now known to be integral.

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In it we considered the genus defined above by Ochanine with values in $\mathbf{Z}[1/2][\delta, \varepsilon]$, regarded as a homomorphism out of the complex cobordism ring MU_*.

Whenever one has an R-valued genus φ on MU_*, one can ask if the functor

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is a cohomology theory, i.e., whether it has appropriate exactness properties.

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LRS, continued

The Ochanine genus $\varphi : MU_* \to \mathbf{Z}[1/2][\delta, \varepsilon]$ does not satisfy these criteria.

We showed that it becomes Landweber exact after inverting either ε or $\delta^2 - \varepsilon$.

This means that if R is the ring obtained from $\mathbf{Z}[1/2][\delta, \varepsilon]$ by inverting one or both of these elements, then the functor

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A homomorphism $\varphi : MU_* \to R$ is also equivalent (by Quillen's theorem) to a 1-dimensional formal group law over R.

When φ is the Ochanine genus, we get the formal group law associated with the elliptic curve defined by the Jacobi quartic,

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LRS and other elliptic curves

The same method entitles us to construct multiplicative homology theories with coefficient rings

$$\mathbf{Z}[\tfrac{1}{6}][g_2, g_3, \Delta^{-1}] \qquad \text{where } \Delta = g_2^3 - 27g_3^2$$

corresponding to the elliptic curve defined by the Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3,$$

and the ring

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LRS and modular forms

$\mathbf{C}[\delta, \varepsilon]$ is naturally isomorphic to the ring $M_*(\Gamma_0(2))$ of modular forms for the group $\Gamma_0(2) \subset \mathrm{SL}_2(\mathbf{Z})$, with δ and ε having weights 2 and 4, respectively.

This isomorphism sends the subring $\mathrm{M}_* = \mathbf{Z}[\frac{1}{2}][\delta, \varepsilon]$ to the modular forms whose q-expansions at the cusp $\tau = \infty$ have coefficients in $\mathbf{Z}[\frac{1}{2}]$.

Moreover, the localizations $\mathrm{M}_*[\Delta^{-1}]$, $\mathrm{M}_*[\varepsilon^{-1}]$ and $\mathrm{M}_*[(\delta^2 - \varepsilon)^{-1}]$ correspond to the rings of modular functions which are holomorphic on $\mathcal{H}$ (the complex upper half plane), $\mathcal{H} \cup \{0\}$ and $\mathcal{H} \cup \{\infty\}$, respectively, and whose q-expansions have coefficients in $\mathbf{Z}[\frac{1}{2}]$.

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The end

Enjoy your retirement, Bob!