THE SEGAL CONJECTURE FOR CYCLIC GROUPS

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The cited conjecture of G. B. Segal (unpublished, c. 1970) concerns the stable homotopy type of BG, the classifying space of the finite group G. In one form it partly describes the stable cohomotopy of BG, asserting that

$$\pi_S^i(BG) = \begin{cases} 0 & \text{if } i > 0 \\ A(G)^\wedge & \text{if } i = 0, \end{cases}$$

where $A(G)^\wedge$ denotes the completion of the Burnside ring of G with respect to the ideal of virtual G-sets of degree 0. The conjecture was proved for $G = Z/(2)$ by W. H. Lin [5], [3] and for $G = Z/(p)$, where p is an odd prime, by J. H. C. Gunawardena [4]. In this note we outline a proof for G cyclic. We will assume that G has prime power order as the general case follows easily.

The proofs cited above make essential use of the Adams spectral sequence [2] and include a hard calculation of a certain Ext group. Our proof uses their results to start an inductive argument. While it does not require any more hard calculations, it does use a certain generalization of the Adams spectral sequence to be described below.

Recall that the function spectrum $F(X, Y)$ by definition represents $[W \wedge X, Y]$ as a functor of W, so $\pi_*(F(X, Y)) = [X, Y]_*$. The functional dual DX of X is $F(X, S^0)$. If X is finite, DX is its Spanier–Whitehead dual and for any X, $\pi_i(DX) = \pi^{-i}(X)$. For a space X let $\Sigma^\infty X_+$ denote the suspension spectrum of X union a disjoint base point. Our main result is

THEOREM 1. *$D\Sigma^\infty BZ/(p^n)_+ = S^0 \vee \bigvee_{i=1}^{n} (\Sigma^\infty BZ/(p^i)_+)_p^\wedge$ where $(\ )_p^\wedge$ denotes the p-adic completion.*

From now on we assume for simplicity that all spectra in sight have been p-adically completed. Let U_i and V_i denote the Thom spectra of the i-fold Whitney sum of the canonical complex line bundles over $BZ/(p^n)$ and $BZ/(p^{n-1})$ respectively. Each is a CW-spectrum with one cell in each dimension $\geqslant 2i$, $U_0 = \Sigma^\infty BZ/(p^n)_+$, and U_{i+1} can be obtained from U_i by collapsing the two bottom cells.

THEOREM 2. *For each $i \geqslant 0$ the composite $U_{-i} \to U_0 \to V_0$ (where the second map is induced by the surjection $Z/(p^n) \to Z/(p^{n-1})$), induces an isomorphism in π^k for $k > 1-2i$.*

We now derive Theorem 1 from Theorem 2. The latter implies that

$$DV_0 \to DU_0 \to DU_{-i}$$

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is a $(2i-2)$-equivalence, so

$$DV_0 \to DU_0 \to \varinjlim DU_{-i}$$

is an equivalence and DV_0 is a retract of DU_0. Let U_i^j denote the j-skeleton of U_i for $j \geqslant 2i$. Then we have a cofibre sequence

$$U_{-i}^{-1} \to U_{-i} \to U_0 .$$

The dual of U_{-i}^{-1} is ΣU_0^{2i-1}, so we have another cofibre sequence

$$DU_0 \to \varinjlim DU_{-i} \to U_0 .$$

We have just seen that the middle spectrum is DV_0 and the first map is a retraction, so

$$DU_0 \simeq DV_0 \vee U_0 .$$

Theorem 1 follows easily by induction on n.

Lin proved Theorem 2 for $n = 1$, $p = 2$ by showing that the composite map induces an isomorphism between the appropriate Adams E_2-terms. Let $P = \varinjlim H^*(U_{-i}) = Z/(2)[x, x^{-1}]$ for $x \in H^1(BZ/(2))$. Since $V_0 = S^0$, one must show that $\mathrm{Ext}_A(Z/(2), P) \simeq \mathrm{Ext}_A(Z/(2), Z(2))$, where A is the Steenrod algebra. A program, which was eventually carried out, for doing this was given by Adams in [1].

For $n > 1$ let $R = \varinjlim H^*(U_{-i})$ (as an A-module this is independent of n if $n > 1$). Using the calculation of Lin and Gunawardena, it is not too hard to show that $\mathrm{Ext}_A(Z/(p), R)$ is a free module over $\mathrm{Ext}_A(Z/(p), Z/(p))$ on generators u_i for $i \geqslant 0$, with $u_i \in \mathrm{Ext}^{[i/2], [i/2]-i}$. On the other hand the Adams spectral sequence for $[V_0, S^0]$ has an E_1-term which is a free module on classes $v_i \in E_1^{0, -i}$ for $i \geqslant 0$; the v_i correspond to the cells in V_0. In order to get the isomorphism needed for Theorem 2 we will modify the spectral sequence for $[V_0, S^0]$ in such a way that the class v_i occurs in $E_2^{[i/2], [i/2]-i}$ and that it maps to u_i.

Recall that the Adams spectral sequence for $[V_0, S^0]$ is based in an Adams resolution for S^0, that is, on a diagram of the form

$$S^0 = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow Y_3 \ldots$$

having certain properties. The maps $Y_j \to Y_{j-1}$ and $V_{i-1} \to V_i$ induce maps

$$F(V_i, Y_j) \to F(V_i, Y_{j-1})$$

and

$$F(V_i, Y_j) \to F(V_{i-1}, Y_j) .$$

These spectra can all be taken to be a subspectra of $F(V_0, S^0)$ and we define $W_s = \bigcup_{0 \leqslant i \leqslant s} F(V_i, Y_{s-i})$.

Our modified Adams spectral sequence for $[V_0, S^0]$ is the one based on the resolution

$$F(V_0, S^0) = W_0 \leftarrow W_1 \leftarrow W_2 \ldots.$$

It is not hard to show that its E_2-term is isomorphic to that of the limit of the usual Adams spectral sequences for $[U_{-i}, S^0]$.

Showing that the latter spectral sequence maps to the former one amounts to showing that the map $U_s \to V_s$ has Adams filtration $\geqslant s$, that is, that it can be written as the composite of s maps each of which is trivial in mod p homology. Recall that $U_s = T(s\lambda)$, the Thom spectrum of $s\lambda$ where λ is the canonical complex line bundle over $BZ/(p^n)$, and that the pullback of the line bundle over $BZ/(p^{n-1})$ is λ^p. Hence the map in question factors through $T(s\lambda^p)$. By an elementary argument the resulting map can be factored as

$$T(s\lambda) \to T\big(\lambda^p \oplus (s-1)\lambda\big) \to T\big(2\lambda^p \oplus (s-2)\lambda\big) \to \ldots \to T\big((s-1)\lambda^p \oplus \lambda\big) \to T(s\lambda^p)$$

with each factor trivial in mod p homology as desired.

The proof of convergence of the two spectral sequences and details of the above arguments will appear elsewhere.

References

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