THE ROOT INVARIANT IN HOMOTOPY THEORY

Mark E. Mahowald
Douglas C. Ravenel

Northwestern University
University of Rochester

For the last thirty years the EHP sequence has been a major conceptual tool in the attempt to understand the homotopy groups of spheres. It is a collection of long exact sequences of homotopy groups induced by certain fibrations in which all three spaces are loop spaces of spheres. These fibrations are due originally to James, G. W. Whitehead, and Toda. The Freudenthal suspension theorem and the Adams vector field theorem (which is a strengthened form of the Hopf invariant one theorem) can each be interpreted as statements about the EHP sequence. James periodicity, the Hopf invariant and the Whitehead product all fit into the EHP framework in a very simple way. An expository survey of this material is given in the last section of the first chapter of [R1].

More recently the work of Morava led the second author and various collaborators to formulate the chromatic approach to stable homotopy theory and the notion of a v_n–periodic family (see [MRW], [R3], [MR] and the last three chapters of [R1]). The recent spectacular work of Devinatz, Hopkins and Smith [DHS] is a vindication of this point of view.

The purpose of this paper is to describe the partial understanding we have reached on how the chromatic and EHP points of view interact. The central concept here is the root invariant, which is defined in 1.10 using Lin's theorem. This assigns to each element in the stable homotopy of a finite complex a nonzero coset in a higher stem. The main conjecture (still unproved) in the subject is that this root invariant converts v_n–periodic families to v_{n+1}–periodic families. The full implications of this are still not understood.

In the first section we will recall the relevant properties of the EHP sequence including James periodicity and define the root invariant in the homotopy of the sphere spectrum. Regular and anomalous elements in the EHP sequence will be defined (1.11).

Both authors were partially supported by the N. S. F.

In the second section we will generalize the definition to finite complexes, describe Jones' connection between the root invariant and the quadratic construction and develop various computational tools. In the third section we will indicate the relation between the root invariant and the Greek letter construction.

In the fourth section we will consider unstable homotopy and define the progeny (4.1) and the target set (4.3) of an element in the EHP sequence. In the fifth section we will prove that an element is anomalous if and only if it is a root invariant. In the last section we will give a method of improving James periodicity in many cases. Finally we will construct some similar spectral sequences in which the theorem connecting anomalous elements and root invariants (1.12) does not hold; these will be parametrized by the p–adic integers.

We thank Hal Sadofsky for pointing out several errors in an earlier version of this paper. We would also like to thank the two referees for the extensive help they have given us.

1. Basic properties of the EHP sequence and the root invariant.

The **EHP spectral sequence** for $p = 2$ is defined from the following filtration of QS^0.

(1.1)
$$\Omega^1 S^1 \to \Omega^2 S^2 \to \cdots \ \Omega^n S^n \to \cdots$$

For odd primes it is necessary to replace the even dimensional spheres by the spaces

$$\hat{S}^{2m} = (\Omega S^{2m+1})^{2m(p-1)}.$$

This is a CW–complex with $p-1$ cells, one in each dimension divisible by $2m$ up to $2m(p-1)$.

James showed that for $p = 2$ there are fibrations

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$$

and Toda showed that for odd primes there are fibrations

$$\hat{S}^{2m} \to \Omega S^{2m+1} \to \Omega S^{2pm+1} \text{ and}$$
$$S^{2m-1} \to \Omega \hat{S}^{2m} \to \Omega S^{2pm-1}.$$

These enable us to identify the relative homotopy groups in the filtration (1.1) and we get the following.

Proposition 1.2. *For $p = 2$ there is a spectral sequence converging to $\pi_*(QS^0)$ with*

(1.3)
$$E_1^{k,n} = \pi_{n+k}(S^{2n-1}).$$

For odd primes we have

$$E_1^{k,2m+1} = \pi_{2m+1+k}(S^{2pm+1}) \text{ and}$$
$$E_1^{k,2m} = \pi_{2m+k}(S^{2pm-1}).$$

In both cases the indexing is such that

$$d_r : E_r^{k,n} \to E_r^{k-1,n-r} \text{ and}$$

$E_\infty^{k,*}$ is the associated graded group for $\pi_k(QS^0)$ filtered by sphere of origin, i.e. by the images of $\pi_k(\Omega^n S^n)$ (with S^{2m} replaced by $\hat{S}^{2m}$ when p is odd).

When

$$\alpha \in \pi_k(QS^0)$$

corresponds to an element

$$\beta \in E_\infty^{k,n},$$

then α desuspends to the n–sphere and we will call β the Hopf invariant of α. We use the equation

$$HI(\alpha) = \beta$$

to represent this connection; β can also be regarded as a coset in E_1.

This spectral sequence is discussed in more detail in Section 1.5 of [R1]. It can be modified so that it converges instead to $\pi_*(\Omega^n S^n)$ (altered as above when p is odd and n is even) by setting $E_1^{k,s} = 0$ for $s > n$. The odd primary homotopy groups of even dimensional spheres can be computed from those of odd dimensional spheres using the odd primary homotopy equivalence

$$\Omega S^{2m} = S^{2m-1} \times \Omega S^{4m-1}.$$

There seem to be several different natural choices for the filtration which defines the spectral sequence. The one we have choosen seems to us to be natural. It is the one which seems treat odd and even primes in a similar fashion. The indices (k, n), have quite natural meaning. In particular, k is the stem as indicated in the statement of the Proposition. The index n is the sphere of origin of the particular class. The identification of (1.3) gives the particular unstable group in which the Hopf invariant lies.

Another discussion of this spectral sequence at the prime 2 and with a point of view very close to what we do in this paper is contained in [M5]. In that paper a different indexing is choosen because only the prime two is considered. It does seem to lead to a nice indexing at odd primes. This paper expands on some of the proofs given there and also extends the results to odd primes.

From now on we will limit most statements to the prime 2 for simplicity.

The first differential,

$$d_1 : E_1^{k,n} \to E_1^{k-1,n-1}$$

is induced by the composite

$$\Omega^3 S^{2n-1} \to \Omega S^{n-1} \to \Omega S^{2n-3}$$

where the factors are maps in the EHP sequence. Gray [G2] shows that if $n = 2m+1$ then this map deloops to give a map

$$\Omega^2 S^{2n-1} \to S^{2n-3}$$

and the composite

$$S^{2n-3} \to \Omega^2 S^{2n-1} \to S^{2n-3}$$

is just the degree two map. If n is even it is conjectured that d_1 is the zero map. It is if $k < 3n-3$, i.e., when $E_1^{k,n}$ is isomorphic to the stable $k-n+1$ stem.

Thus $E_2^{k,n}$ is a $Z/2$ vector space for $n > 1$ and $k < 3n-3$. A sharper analysis in the unstable range ($k \geq 3n-3$) by James [J1] shows that $E_2^{k,n}$ is always a $Z/2$–vector space and this leads to the bounds on the torsion in $\pi_*(S^n)$ proved by James. Toda proved similar results for odd primes. Thus in dealing with the EHP spectral sequence we never know classes any better than in this associated graded sense. Hence in the following when we refer to an element we will always be discussing the coset of the class modulo twice the class. It is not possible to give distinguished generators in this setting. This raises some interesting questions when dealing with classes such as generators in the image of the J–homomorphism which do have a distinguished generators.

We will sumarize this information in the following proposition.

Proposition 1.4. *Let*

$$\alpha \in E_1^{k,2m+1}$$

correspond under the identification 1.3 to an element β which is the double suspension of some β'. Then $d_1(\alpha)$ corresponds to $2\beta'$.

The Freudenthal suspension theorem gives

$$E_1^{k,n} = E_1^{k+1,n+1}$$

for $k < 3n-3$ for $p = 2$. There is a similar result for an odd prime p in which the upper bound on k is roughly $(p^2-1)n$. This part of the E_1-term is called the **metastable range.** It is called the **stable zone** in [R1]. The following is a consequence of Toda's [T2] and James' [J2].

Proposition 1.5. *Let $(k-1, n-r+1)$ be in the metastable range. Then for each prime p there is a function $f(r)$ such that for each r there is an isomorphism*

$$E_r^{k,n} = E_r^{k+(p-1)f(r),n+f(r)}.$$

This is called **James periodicity**. For $p = 2$ the function $f(r)$ is related to the vector field number and is given by

$$f(r) = 2^{g(r)} \text{ where } g(r) = [r/2] + \begin{cases} -1 & \text{if } r \equiv 0 \mod 8 \\ 0 & \text{if } r \equiv 1 \\ 0 & \text{if } r \equiv 2 \\ 1 & \text{if } r \equiv 3 \\ 0 & \text{if } r \equiv 4 \\ 1 & \text{if } r \equiv 5 \\ 0 & \text{if } r \equiv 6 \\ 0 & \text{if } r \equiv 7. \end{cases}$$

For $p > 2$ we have

$$f(r) = 2p^{[r/2]}.$$

Proof. A class

$$a \in E_1^{k,n} = \pi_k(\Omega^n S^{2n-1})$$

is an r–cycle if and only if it maps to zero under the composite

$$\pi_k(\Omega^n S^{2n-1}) \to \pi_{k-1}(\Omega^{n-1} S^{n-1}) \to \pi_{k-1}(\Omega^{n-1} S^{n-1}, \Omega^{n-r-1} S^{n-r-1}).$$

Toda [T1] has shown

$$\pi_{k-1}(\Omega^{n-1} S^{n-1}, \Omega^{n-r-1} S^{n-r-1}) = \pi_{k-1}^S(P_{n-r-1}^{n-2})$$

for $k-1 < 3(n-r-1)-3$. The above composition is just the map induced by

$$p : S^{n-2} \to P_{n-r-1}^{n-2}$$

where p is the attaching map of the $(n-1)$–cell. James [J2] has shown $P_{n-r-1}^{n-1} \cong \Sigma^{f(r)} P_{n-r-1-f(r)}^{n-1-f(r)}$ for $f(r)$ as above.

Alternativley, consider the following commutative diagram.

$$\begin{array}{ccccc} \Omega^n S^n & \longrightarrow & \Omega^{n+1} S^{n+1} & \longrightarrow & \Omega^{n+1} S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ QRP^{n-1} & \longrightarrow & QRP^n & \longrightarrow & QS^n \end{array}$$

The vertical maps here are Snaith maps. The EHP spectral sequence is based on the exact couple in homotopy derived from the first row. The Snaith maps induce a map to the spectral sequence associated with the bottom row, which is (up to reindexing) the Atiyah–Hirzebruch spectral sequence for $\pi_*^S(RP^\infty)$.

Toda's isomorphism follows from the fact that the right vertical map induces an isomorphism in homotopy in the relevant range of dimensions. The periodic behavior of differentials in the second spectral sequence follows from James' result about the stable homotopy type of stunted projective spaces. □

The function $f(r)$ above is based on the homotopy type of certain stunted real projective spaces. For odd primes the stunted projective spaces have to be replaced by stunted forms of $B\Sigma_p$, the classifying space of the symmetric group on p letters, localized at p. This is a CW–complex with one cell in each dimension congruent to 0 or $-1 \mod 2p-2$.

Experience has shown that many differentials in the EHP spectral sequence in fact have shorter periods than predicted above. In Section 5 we will discuss methods for improving this result in certain cases.

This result can be used to define a new spectral sequence, the *stable EHP spectral sequence* $\{SE_r^{k,n}\}$. (This was called the superstable EHP spectral sequence in [R1].)

Definition 1.6. *For each integer n,*

$$SE_r^{k,n} = E_r^{k',n'}$$

where k' and n' are chosen so that

$$0 < k' < 3n' - 3, k \equiv k' \mod f(r) \text{ and } n \equiv n' \mod f(r).$$

In particular we have

$$SE_1^{k,n} = \pi_{k+1-n}(QS^0). \tag{1.7}$$

The differential (of period $2f(r)$) on $SE_r^{k,n}$ is determined by that on $E_r^{k',n'}$ with $k \equiv k'$ and $n \equiv n' \mod 2f(r)$.

It is known that in the metastable range this spectral sequence and the EHP spectral sequence coincide with a suitably reindexed form of the Atiyah–Hirzebruch spectral sequence converging to the stable homotopy of $B\Sigma_p$. For $p = 2$ the reindexing is such that our nth row $E_1^{*,n}$ corresponds to the $(n-1)$-cell in RP^∞.

For $p = 2$ this can be deduced from the diagram use din the proof of 1.5.

Now consider the Atiyah–Hirzebruch spectral sequence used to calculate $\pi_*(P_k)$. Here P_k for $k \geq 0$ denotes (for $p = 2$) the stunted real projective space RP^∞/RP^{k-1}. P_k can be defined for $k < 0$ as a suitable Thom spectrum or via James periodicity. There is a pinch map from P_k to P_{k+1}. This leads to the inverse system:

$$\pi_*(\text{holim } P_{-k}) = \lim_{\leftarrow} \pi_*(P_{-k}).$$

When working at an odd prime we will let P_i^{i+k} denote the stunted form of $B\Sigma_p$ localized at p, with bottom cell in dimension i and top cell in dimension $i+k$. These two numbers must be congruent to 0 or -1 modulo $2p-2$ since $B\Sigma_p$ has no cells in other dimensions.

Let $\{LE_r^{k,n}\}$, with $k \in Z$, be this spectral sequence reindexed as above, i.e., the reindexed Atiyah–Hirzebruch spectral sequence for $\pi_*(\text{holim } P_{-k})$. Then the following result was proved in [M5].

Proposition 1.8.

(a) *The spectral sequences* $\{SE_r^{k,n}\}$ *and* $\{LE_r^{k,n}\}$ *coincide and*

(b) [L] *they converge to the 2–adic completion of* $\pi_*(S^{-1})$.

The first statement is an easy observation while the second is a reformulation of Lin's theorem. An odd primary analog of the latter can be found in [AGM]. In it stunted real projective spaces are replaced by similar spectra associated with the p–component of $B\Sigma_p$, the classifying space for the symmetric group on p letters.

Another approach to this spectral sequence is obtained by considering the EHP sequence in the neighborhood of $k = 2^i$ for large i. In particular, let $\bar{E}_1^{k,s}(i,j) = 0$ for $s > 2^i + j$ or $s \leq 2^i - j$ and $\bar{E}_1^{k,s}(i,j) = E_1^{k,s}$ otherwise. Then

$$\bar{E}_r^{k,*}(i,j) \Rightarrow \pi_k(\Omega^{2^i+j}S^{2^i+j}, \Omega^{2^i-j}S^{2^i-j}).$$

Toda's result used in the proof of 1.5 gives

$$\bar{E}_r^{k,*}(i,j) \Rightarrow \pi_k^S(P_{2^i-j}^{2^i+j-1})$$

for $k < 3(2^i - j) - 3$. For large i and h, James periodicity gives

$$\Sigma^{2^i} P_{2^h-j}^{2^h+j-1} \approx \Sigma^{2^h} P_{2^i-j}^{2^i+j-1}.$$

Thus we can define $\bar{L}E_r^{k,n} = \bar{E}_r^{k+2^i,n+2^i}(i,j)$ for large i and j chosen so that $r - j < k < j - r$ and $r - j < n < j - r$. We then have:

Proposition 1.9. *The spectral sequences* $\{SE_r^{k,n}\}$ *and* $\{\bar{L}E_r^{k,n}\}$ *coincide.*

Definition 1.10. *Let*

$$\alpha \in \pi_*(S_{\hat{2}}^{-1})$$

correspond to

$$\beta \in SE_\infty^{k,n}$$

The **root invariant** $R(\alpha)$ *is the corresponding coset in* $\pi_{k+1-n}(QS^0)$ *under the identification (1.7). (Note that the group* $SE_\infty^{k,n}$ *is a subquotient of* $SE_1^{k,n}$ *which is isomorphic to* $\pi_{k+1-n}(QS^0)$*.)*

A more geometric definition is as follows. Let α be represented by a map

$$f : S^{t-1} \to S^{-1}.$$

Composition with the inclusion

$$i : S^{-1} \to P_{-1}$$

is null homotopic by the Kahn–Priddy theorem [KP], provided that f is not a unit multiple of the identity map. To see this note that in the cofibration

$$\Sigma^{-1} P_0 \to S^{-1} \to P_{-1}$$

the lefthand map is the one studied by Kahn–Priddy. They show that the cokernel of the induced map in homotopy is $Z/2$ generated by the fundamental class.

The inclusion of S^{-1} into P_{-1} lifts to P_{-n} for all $n > 0$. One way to see this is to note that the Thom spectrum of the normal bundle over RP^{n-1} is a suspension of P_{-n}^{-1}. The top cell in any normal Thom spectrum is spherical and the composite

$$S^{-1} \to P_{-n}^{-1} \to P_{-n}$$

is the desired lifting, which we will denote by h.

Since

$$\text{holim } P_n = S_{\hat{2}}^{-1}$$

by Lin's theorem, there is an n such that the composite

$$S^{t-1} \to S^{-1} \to P_{-n}$$

is essential. Choose the smallest such n. Then the composite

$$S^{t-1} \to P_{-n} \to P_{1-n}$$

is null homotopic so we get a (not necessarily unique) map

$$g : S^{t-1} \to S^{-n}.$$

The coset (in $\pi_{n+t-1}(S^0)$) of all such maps is the root invariant of α.

Proposition 1.9 gives another way to view the root invariant. If j is smaller than the number of vector fields on S^{2^i-1},then there is a class ι_{2^i-1} in $\pi_{2^i-1}(\Omega^{2^i+j}S^{2^i+j}, \Omega^{2^i-j}S^{2^i-j})$ which is detected in mod 2 homology. Let $\alpha \in \pi_k(S^0)$, $k < j$. Suppose also that the class $\alpha\iota \neq 0$. Then one of two things happens:

(1) $\alpha\iota$ is in the image of the map

$$\pi_{k+2^i-1}(\Omega^{2^i+j}S^{2^i+j}) \to \pi_{k+2^i-1}(\Omega^{2^i+j}S^{2^i+j}, \Omega^{2^i-j}S^{2^i-j}).$$

Then $R(\alpha) = HI(\alpha\iota)$.

(2) Case (1) does not occur. Then $\partial_*(\alpha\iota) \in \pi_{k+2^i-2}(\Omega^{2^i-j}S^{2^i-j})$ is nontrivial but unstable, since its image in $\pi_{k+2^i-2}(\Omega^{2^i+j}S^{2^i+j})$ is trivial. Then for some m between j and $-j$, it is $[\iota_{2^i-m}, \beta]$ for some β. (Any unstable element can be written in this way.) Then $R(\alpha) = \beta$.

Jones has shown in [Jo] that the root invariant is related to the quadratic construction. In particular the dimension of $R(\alpha)$ is at least twice that of α. (See Theorem 2.1 below.)

Now we return to the EHP spectral sequence. Recall that if $t < 2n - 2$ then

$$E_1^{t+n-1,n} = \pi_{t+n-1}(\Omega^n S^{2n-1}) \simeq \pi_t(S^0).$$

Definition 1.11. *For a stable element $x \in \pi_t(S^0)$, let $\phi_n(x)$ denote the element in $E_1^{t+n-1,n}$ corresponding to x. (This is the appropriate bigrading for $p = 2$. For $p > 2$, $\phi_n(x)$ is the element born on S^n with Hopf invariant x.) A stable homotopy element x as above is* **regular** *if there is an r such that $\phi_n(x)$ is either zero or not present as a cycle in $E_r^{k,n}$ for all but finitely many n. If α is not regular it is* **anomalous***.*

If $x = \nu$, then $\phi_n(x)$ is nontrivial permanent cycle if $n = 2^j - 2$ and $j \geq 3$. (These are the elements η_j of [M3].) There are many other examples. One of the main results of this paper is the following.

Theorem 1.12. *A class $x \in \pi_t(S^0)$ is anomalous if and only if it is a root invariant.*

It is easy to see that a root invariant is anomalous. If x is a root invariant, then for some $n, \phi_n(x)$ is a nontrivial permanent cycle in the stable EHP spectral sequence. For each r, James periodicity tells us that $\phi_m(x)$ is a nontrivial r–cycle for each m congruent to n modulo a suitable power of 2. Thus x is anomalous by definition. The converse is more delicate, and will be proved in Section 5.

Theorem 1.12 leads to some insight about the general behavior of the EHP sequence. Suppose that $R(\alpha) = \beta$. This means that in the stable EHP spectral sequence, $\phi_0(\alpha)$ is the target of a differential (the Kahn-Priddy theorem) and that the element

$$\alpha\iota_{-1} \in \pi_*(S^{-1})$$

corresponds to $\phi_t(\beta)$ (which is a nontrivial permanent cycle) for a suitable negative value of t. It follows from James periodicity (1.5) that among those $\phi_s(\beta)$ which support nontrivial differentials, *the length of those differentials increase as s approaches t p–adically.* If we write

$$d_r(\phi_s(\beta)) = \phi_{s-r}(\gamma_s) \neq 0,$$

(where r depends on s) we conclude that there are infinitely many distinct γ_s. The set of such targets is studied in Section 4.

One might ask how common root invariants are. We believe they are quite rare. We can offer the following empirical observations. By Jones' Theorem (2.1 below), the dimension of $R(\alpha)$ is at least p times that of α and experience shows that it is less than $p+1$ times that amount, assuming the dimension of α is positive. Usually the ratio of the two dimensions is very close to p. We also know empirically that the size of the stable k–stem (by which we mean the base p logarithm of the order of its p–component) grows *linearly* with k. This suggests the following.

Conjecture 1.13. *Let G_k denote the p–component of the stable k–stem and $R_k \subset G_k$ the subgroup generated by root invariants. Let $|G_k|$ denote the base p logarithm of the order of G_k and similarly for $|R_k|$. Then*

$$\lim_{k\to\infty} \frac{|R_k|}{|G_k|} = \frac{1}{p^2}.$$

2. Computing the root invariant

First observe that Proposition 1.4 tells us that if α is divisible by 2 and has order divisible by 4 then α is regular and the r in Definition 1.11 is 2. In particular, this means that only generators or elements of order 2 can be root invariants (or Hopf invariants of stable elements). Because of this we will identify a class α with the coset $\alpha \mod 2\alpha$.

The root invariant of the fundamental class ι is ι, but in every other case the Kahn–Priddy theorem [KP] implies that $R(\alpha)$ lies in a higher stem than α. Jones [Jo] has done better. Suppose $f : S^{n+t} \to S^n$ represents α. This extends to a map

$$Q_N(f)' : \Omega^N \Sigma^N S^{n+t} \to \Omega^N \Sigma^N S^n.$$

This defines, using the Snaith splitting [S] of $\Omega^N \Sigma^N$, a map

$$Q_N(f)'' : \Sigma^{n+t} P_{n+t}^{N+n+t-1} \to \Sigma^n P_n^{N+n-1}.$$

We can compose this with the pinching map to get

$$\Sigma^{n+t} P_{n-t-1}^{N+n+t-1} \to \Sigma^{n+t} P_{n+t}^{N+n+t-1} \to \Sigma^n P_n^{N+n-1}.$$

If n is divisible by a sufficiently large power of 2 then the target reduces to S^{2n}, i.e., its bottom cell splits off and we get

$$\Sigma^{n+t} P_{n-t-1}^{N+n+t-1} \to S^{2n}$$

which we denote by $Q_N(f)$ and call the *quadratic construction* on f. (The additional cells in the source are necessary as consideration of the map η will indicate.)

Theorem 2.1 [Jo]. *If f is essential then so is $Q_N(f)$ for some N. Furthermore if N is the smallest integer such that $Q_N(f)$ is essential then the latter factors*

$$\Sigma^{n+t} P_{n-t-1}^{n+t+N-1} \to S^{2n+2t+N-1} \to S^{2n}$$

where the second map represents $R(\alpha)$.

Corollary 2.2. *The dimension of $R(\alpha)$ is at least twice that of α.*

Results which predate Lin's theorem and which shed light on the root invariant particularly through Jones' theorem first appear in a paper of Milgram [Mi1]. Milgram constructs squaring operations in $\mathrm{Ext}_A(\mathbf{Z}/(2), \mathbf{Z}/(2))$, the E_2–term of the classical Adams spectral sequence for the sphere spectrum. (We will often abbreviate this by Ext.) The construction is very similar to Jones'. The cells in the projective space correspond to $\cup_i$ constructions. In $\mathrm{Ext}^{s,t}$ they double the t value and increase the s value by $2s-i$. They are not defined if $i > s$. If $i = 0$, the operation is just the square of the class. As usual, if $a \in \mathrm{Ext}^s$ we will write $\mathrm{Sq}^i(a)$ for $a \cup_{s-i} a$. In this setting Sq^0 is not the identity but, in general, is a non-zero class in twice the t-filtration but the same s-filtration. There are several interesting things to notice. First, for each $a \in \mathrm{Ext}^{s,t}$ the class $\mathrm{Sq}^{s-i}(a) = a \cup_i a \in \mathrm{Ext}^{2s-i,2t}$ is defined for all $i \leq s$. This should be compared with the homotopy version of $\alpha \cup_i \alpha$ which is defined usually only if $\alpha \cup_{i'} \alpha = 0$ for all $i' < i$. Second, notice that $a \cup_i a$ is not defined for $i > s$ although Jones needed, in general, this case. This filtered version of the quadratic construction is exploited extensively by Bruner [B]. It is interesting to note that nearly all the differentials in the Adams spectral sequence known to date are a consequence of the quadratic construction understood this way. In homotopy, the $\cup_0$ is the first one which one sees and this is the statement of Corollary 2.2. In the Ext setting $\cup_s$ is the only Squaring operation which one sees. Often this is quite far from the actual root invariant. We will return to this point later in this section.

A proof of the odd primary analog of Jones' theorem has recently been given by Haynes Miller in [Mil].

The following is an easy calculation from Definition 1.10. The $p = 2$ results are interpretations of calculations made in [M6].

Proposition 2.3. *$R(p\iota) = \alpha_1$ where*

$$\alpha_1 \in \pi_{2p-3}(S^0)$$

is the first element of order p. This element also has Hopf invariant 1. For $p = 2$ this is the Hopf map η. Moreover we have

$$R(\eta) = \nu,$$
$$R(\nu) = \sigma \text{ and}$$
$$R(\sigma) = \sigma^2$$

where ν and σ are the Hopf maps in the 3– and the 7–stems.

For odd primes we will show below (2.13 and 2.15) that

$$R(\alpha_1) = \beta_1 \text{ and}$$
$$R(\beta_1) = \beta_1^p$$

where β_1 is the first element not in the image of the J–homomorphism, in the $(pq-2)$–stem.

The following is another easy consequence of Jones' theorem, using results from [BJM §4].

Proposition 2.4. *If*

$$\theta_j \in \pi_n(S^0)$$

with $n = 2^{j+1} - 2$ *corresponds to the element* h_j^2 *in the Adams* E_2*–term and* $\theta_j^2 = 0$*, then*

$$\theta_j \in R^{j+1}(2\iota) \text{ for } j \geq 4.$$

These are the celebrated Kervaire invariant elements, known to exist only for $j \leq 5$. For $p > 2$ there are analogous elements b_j for $j \geq 0$ of Adams filtration 2 which were studied in [R2] (also in section 6.4 of [R1]). They correspond roughly to elements in the Adams–Novikov E_2–term

$$\beta_{p^j/p^j} \in E_2^{2p^{j+1}(p+1)}$$

in [R2] it was shown that for $p \geq 3$ only the first of these (which is β_1) is a permanent cycle.

For all primes, the root invariant defines an interesting infinite set of elements

$$\{R^j(p\iota)\}.$$

At $p = 2$ we can identify these elements under some additional hypothesis as the Kervaire invariant one elements. Even if there are only a finite number of Kervaire invariant one elements this set is of considerable interest. The work in [R2] concerning the "odd primary Kervaire invariant elements" is related to identifying these elements at odd primes. There still remains "a Kervaire invariant" problem at all primes, i.e., identify the cosets $R^j(p\iota)$. The odd primary analog of Jones' theorem shows that the dimensions of these elements must grow exponentially. The problem then is not to show that these are homotopy elements, but to identify them and to determine the stems in which they lie.

It is possible to mimic the definition of the root invariant in various algebraic approximations to stable homotopy such as the Λ–algebra, the Adams E_2–term and presumably the Adams–Novikov E_2–term. We will discuss some calculations of these below. The definition in the Adams E_2–term is as follows.

We need to compute in

$$\mathrm{Ext}_A(H^*(P_{-n}),\ \mathbf{Z}/(2))$$

for various n. We will abbreviate this group by

$$E_2(P_{-n}).$$

This is, of course, the E_2–term of the Adams spectral sequence for P_{-n}.

The inclusion i of the bottom cell in P_{-1} induces a map

$$i_{\#} : E_2(S^{-1}) \to E_2(P_{-1}).$$

(We will use the subscript # to denote induced maps of E_2.) There is an algebraic analog of the Kahn–Priddy theorem proved by Lin [L2] which implies that this map annihilates all but the fundamental class. Given a class $x \in E_2(S^0)$ suppose its image under this map is zero. There are maps

$$E_2(S^{-1}) \to E_2(P_{-n}).$$

for each $n > 0$. We need to know that x must have a nonzero image if n is sufficiently large. The main result of [LDMA] is that

$$\lim_{\leftarrow} E_2(P_{-n}) = E_2(S^{-1}).$$

This implies that x has a nonzero image for some n. We pick the smallest such n and proceed as in Definition 1.10.

Above we discussed the connection between the quadratic construction and $\cup_i$ constructions. The only $\cup_i$ construction on a class $a \in \mathrm{Ext}_A^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ which is filtration preserving is $\cup_s$ which is represented as Sq^0. This establishes the following as a corollary of Jones' theorem.

Proposition 2.5. *In the Adams E_2–term one has*

$$h_{j+1} \in R(h_j) \text{ and}$$
$$h_{j+1}^2 \in R(h_j^2).$$

More generally, if $\mathrm{Sq}^0(a) \neq 0$ *then* $\mathrm{Sq}^0(a) \in R(a)$.

This connection with squaring operations suggests that there should be a Cartan formula for root invariants. We have not been able to prove a nice one. The following gives a weak connection between the root invariant of a product and the root invariants of the factors.

Corollary 2.6. *If a and b are two classes in $E_2(S^0)$ with*

$$\mathrm{Sq}^0(a)\mathrm{Sq}^0(b) \neq 0,$$

then

$$\mathrm{Sq}^0(a)\mathrm{Sq}^0(b) \in R(a)R(b) \subset R(ab).$$

The relation between algebraic and homotopy root invariants is complicated by the usual problems which arise from associated graded answers which result from spectral sequences. It is quite easy to lose the answer in the resulting indeterminacy. We will discuss the situation in the classical Adams spectral sequence. Something similar is valid in the Adams-Novikov spectral sequence.

Consider the commutative diagram of Ext groups

$$\begin{array}{ccccc} E_2(S^{-n}) & \xrightarrow{i_\#} & E_2(P_{-n}) & \xrightarrow{j_\#} & E_2(P_{-n+1}) \\ {\scriptstyle v}\uparrow & & {\scriptstyle h_\#}\uparrow & & \\ E_2(S^{-1+t}) & \xrightarrow{u} & E_2(S^{-1}) & & \end{array} \tag{2.7}$$

We use the subscript # to indicate maps induced in Ext groups and use $*$ to indicate maps induced in homotopy. u denotes the homomorphism which sends the fundamental class in $E_2(S^{-1+t})$ to the class $u \in E_2(S^{-1})$ and similarly for v. i and h are the usual maps and j is the pinch map. The integer n is chosen to be the smallest such that $h_\# u$ is nontrivial. Then v is in the algebraic root invariant coset $R(u)$.

We will assume that u is a nontrivial permanent cycle and is represented by a map f with $u = f_\#$. (This f is of course not uniquely determined by u, but usually in practice we are given f, can compute $R(u)$, and want to deduce something about $R(f)$.) Even with this assumption, $R(u)$ may fail to contain a permanent cycle.

Suppose n is not the smallest integer such that hf (as opposed to $h_\# f_\#$) is nontrivial. This can happen when jhf is essential but has higher Adams filtration than expected. Then there is no commutative diagram of the form

$$\begin{array}{ccccc} S^{-n} & \xrightarrow{i} & P_{-n} & \xrightarrow{j} & P_{-n+1} \\ {\scriptstyle g}\uparrow & & {\scriptstyle h}\uparrow & & \\ S^{-1+t} & \xrightarrow{f} & S^{-1}. & & \end{array} \tag{2.8}$$

and $R(u)$ does not contain a permanent cycle, *and the homotopy root invariant $R(f)$ is in a smaller stem than the algebraic root invariant $R(u)$.*

On the other hand, if n is the smallest integer such that hf is essential, then jhf is null, (2.8) exists, and *$R(f)$ lies in the same stem as $R(u)$.* However it is still possible that $h_\#(u)$ is the target of a differential. *Then the map $hf = gi$ and hence the coset $R(f)$ have higher Adams filtration than expected.*

If $h_\#(u)$ is nontrivial in E_∞ and jhf is null, then $R(u)$ contains a permanent cycle represented by g. In this case one can ask about the relation between the homotopy coset $R(f)$ and the the homotopy coset $\{R(u)\}$ corresponding to the Ext coset $R(u)$. Each choice of g in (2.8) represents some choice of v in (2.7), so we have

$$R(f) \subset \{R(u)\}.$$

Finally, if hf is null (which could happen if $h_{\#}(u)$ is the target of a differential), then *the homotopy root invariant $R(f)$ is in a larger stem than the algebraic root invariant $R(u)$.*

We summarize with the following theorem.

Theorem 2.9. *Let $\alpha \in \pi_t(S^0)$ be a nontrivial homotopy element representing a class $u \in E_2(S^0)$ and let the algebraic root invariant coset $R(u)$ be in dimension k.*

(1) *If $R(u)$ does not contain a permanent cycle then the dimension of $R(\alpha)$ is less than k.*
(2) *If there is a diagram like (2.8) above but $h_{\#}(u)$ is killed by a differential, then $R(\alpha)$ has the same stem but higher Adams filtration than $R(u)$.*
(3) *If (2.8) exists and the map $h_{\#}(u)$ is nontrivial in E_∞, then $R(\alpha)$ is contained in the homotopy coset representing $R(u)$.*
(4) *If (2.8) exists and the map hf is null, then the dimension of $R(\alpha)$ is in a greater than k.*

Notice that we are not claiming that if $R(u)$ contains a permanent cycle then (2.8) exists.

The definition of the root invariant can be extended to the homotopy of a finite complex X in a straightforward way.

Definition 2.10. *Let X be a p–local finite spectrum and let*

$$\alpha \in \pi_*(\Sigma^{-1}X)$$

correspond to

$$\beta \in LE_\infty^{k,n}$$

where $LE_r^{k,n}$ is the reindexed Atiyah–Hirzebruch spectral sequence for $\pi_*(X \wedge P_{-n})$. Since smashing with a finite spectrum commutes with homotopy inverse limits,

$$X_*(\text{holim } P_{-n}) = \varprojlim \pi_*(X \wedge P_{-n})$$

and the root invariant $R(\alpha)$ is the corresponding coset in $\pi_{k+1-n}(X)$.

Lin's theorem implies that for any p–adically complete finite complex X,

$$\text{holim } (P_{-k} \wedge X) = \Sigma^{-1}X.$$

This means that we always get a nontrivial invariant.

Recall that we needed the Kahn–Priddy theorem to show that $R(\alpha)$ is not α. This theorem does not generalize to a finite complex, but we do have

Lemma 2.11. *Let X be a p–local finite complex. Then the map*

$$\Sigma^{-1}X \to X \wedge P_{-1}$$

has a finite image in homotopy.

Proof. Kahn–Priddy prove their theorem by showing that the space Q_0S^0 is a factor of $QB\Sigma_p$. Let Y be a Spanier–Whitehead dual of X desuspended as far as possible. Then it follows that the group

$$[\Sigma^*Y, Q_0S^0] = \pi_*(X)$$

is a summand of

$$[\Sigma^*Y, QRP^\infty] = \pi_*(X \wedge P_1).$$

The trouble is we lose some of the lower homotopy groups because we cannot necessarily desuspend Y as far as we would like. It follows that the map

$$\pi_*(X) \to \pi_{*-1}(X \wedge P_{-1})$$

is zero above a certain range depending on X. $\square$

A useful variation of the root invariant is the following. Suppose we replace the cofibre sequence

$$S^{-n} \to P_{-n} \to P_{1-n}$$

by

$$\Sigma^{-1-qm}V(0) \to P_{-1-qm} \to P_{q-1-qm}$$

where $V(0)$ denotes the mod p Moore spectrum and $q = 2p-2$. Given $f : S^{t-1} \to S^{-1}$ we look for the smallest n such that the composite

$$S^{t-1} \xrightarrow{f} S^{-1} \xrightarrow{h} P_{-1-qm}$$

is essential and thereby produce a (not necessarily unique) map $g : S_{t-1} \to \Sigma^{-1-qm}V(0)$.

Definition 2.12. *If $\alpha \in \pi_t(S^0)$ is the class of the map f above, then the* **modified root invariant** $R'(\alpha)$ *is the coset of $\pi_{qm+t}(V(0))$ containing the map g constructed above.*

The relation of $R'(\alpha)$ to $R(\alpha)$ is transparent.

Proposition 2.13. *Suppose $R(\alpha) \subset \pi_{t+n}(S^0)$ for $\alpha \in \pi_t(S^0)$. Then n must have the form qm or $qm - 1$. In the former case $R'(\alpha)$ is the mod p reduction of $R(\alpha)$, which is necessarily nontrivial since $R(\alpha)$ has a nontrivial image in $\pi_{t-1}(P_{-1-qm})$. In the latter case $R'(\alpha)$ maps to $R(\alpha)$ under the pinch map $j : V(0) \to S^1$. Conversely if $R'(\alpha)$ has a nontrivial image under the pinch map then $n = qm - 1$ and that image is $R(\alpha)$. If the image is trivial then $n = qm$ and $R(\alpha)$ is the preimage of $R'(\alpha)$ under the reduction map $S^0 \to V(0)$.*

Another useful notion for computations of root invariants is the following.

Definition 2.14. . *Let E be a ring spectrum and let $\alpha \in \pi_t(X)$ where X is a p–local finite spectrum. We say α has a nontrivial E_*–root invariant if the composite*

$$S^{t-1} \to \Sigma^{-1} X \to \operatorname{holim}\,(P_{-n} \wedge E \wedge X)$$

is essential. We denote the resulting coset in $E_*(X)$ by $R_E(\alpha)$. The modified E_*–root invariant $R'_E(\alpha) \subset E_*(V(0))$ is similarly defined.

Theorem 2.15. *If $R_E(\alpha)$ is a coset in $\pi_*(E)$ containing a class in the image of a class $\beta \in \pi_*(S^0)$ such that the following diagram commutes*

$$\begin{array}{ccc} S^{-1} & \xrightarrow{h} & P_{-n} \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle i} \\ S^{t-1} & \xrightarrow{\beta} & S^{-n}, \end{array}$$

then $\beta \in R(\alpha)$. Similarly if $R'_E(\alpha)$ is a coset in $\pi_*(V(0) \wedge E)$ containing a class in the image of a class $\beta' \in \pi_*(V(0))$ making a similar diagram commute, then $\beta' \in R'(\alpha)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} & & & & P_{1-n} & \longrightarrow & P_{1-n} \wedge E \\ & & & & \uparrow & & \uparrow \\ S^{t-1} & \xrightarrow{\alpha} & S^{-1} & \xrightarrow{h} & P_{-n} & \xrightarrow{\eta} & P_{-n} \wedge E \\ & & & & \uparrow & & \uparrow \\ & & & & S^{-1} & \longrightarrow & \Sigma^{-n} E. \end{array}$$

where each column is a cofibre sequence. By assumption, $\eta h \alpha$ is essential (and therefore $h\alpha$ is) and it lifts to a map β to S^{-n}. The result follows about the root invariant follows. The argument for the modified root invariant is similar. □

Now we want to compute $R_{bo}(2^i\iota)$ and $R'_{bo}(2^i\iota)$. We need to recall the structure of $\pi_*(bo)$ and $\pi_*(V(0)\wedge bo)$. We have

$$\pi_*(bo) = \mathbf{Z}[\eta,\alpha,\beta]/(2\eta,\eta^3,\eta\alpha,\alpha^2-4\beta)$$

with $\eta\in\pi_1$, $\alpha\in\pi_4$ and $\beta\in\pi_8$, so

$$\pi_i(bo) = \begin{cases} \mathbf{Z} & \text{if } i\equiv 0 \mod 4 \\ \mathbf{Z}/(2) & \text{if } i\equiv 1 \text{ or } 2 \mod 8 \\ 0 & \text{otherwise.} \end{cases}$$

$\pi_*(V(0)\wedge bo)$ is a module over $\pi_*(bo)$ on generators $\iota\in\pi_0$ and $v\in\pi_2$ with relations $2\iota=0$, $2v=\eta^2\iota$ and $v\eta^2=\alpha$, so

$$\pi_i(V(0)\wedge bo) = \begin{cases} \mathbf{Z}/(2) & \text{if } i\equiv 0,1,3 \text{ or } 4 \mod 8 \\ \mathbf{Z}/(4) & \text{if } i\equiv 2 \mod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.16. *For all primes p, p^i has a nontrivial bo_*–root invariant, where bo is the spectrum for real connective K–theory. (For an odd prime p we can replace bo by $BP\langle 1\rangle$, which is a summand of bo.) For $p>2$ the dimension of both $R_{bo}(p^i)$ and $R'_{bo}(p^i)$ for $i\geq 0$ is qi, where $q=2p-2$. For $p=2$ we have*

$$R_{bo}(2^{4a+b}) = \begin{cases} \beta^a & \textit{if} \quad b=0 \\ \eta\beta^a & \textit{if} \quad b=1 \\ \eta^2\beta^a & \textit{if} \quad b=2 \\ \alpha\beta^a & \textit{if} \quad b=3 \end{cases}$$

(in other words $R_{bo}(2^i\iota)$ is the ith generator in $\pi_*(bo)$) and

$$R'_{bo}(2^{4a+b}) = \begin{cases} \beta^a & \textit{if} \quad b=0 \\ v\beta^a & \textit{if} \quad b=1 \\ \eta^2\beta^a & \textit{if} \quad b=2 \\ \alpha\beta^a & \textit{if} \quad b=3. \end{cases}$$

Proof. It is easy to calculate $\pi_*(P_n\wedge bo)$; the relevant Adams spectral sequence collapses (see [M4] Section 7). It follows that

$$\pi_i \lim_{\leftarrow}(P_{-n}\wedge bo) = \begin{cases} \mathbf{Z}_2 & \text{if } i\equiv 3 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{Z}_2$ denotes the 2–adic integers. The Adams filtration of each element in this group is determined by its divisibility by two.

This group could also be calculated by a spectral sequence based on the skeletal filtration of P_{-n}. The element in π_{-1} which is 2^i times the generator has the form $R_{bo}(2^i)x_j$ where $j-1=\dim R_{bo}(2^i)$ and x_j is the class corresponding to the j–cell in P_{-n}.

The E_2–term of this spectral sequence has subquotients of the form

$$\pi_*(P_{2m-1}^{2m} \wedge bo).$$

This calculation leads to the stated result for $p = 2$. The odd primary case is similar and easier. □

Now we want to compute $R(p^i\iota)$.

Theorem 2.17. *For all $i > 0$ and all primes p, $R(p^i\iota)$ is the first positive dimensional element in the Adams E_2–term of filtration i. In other words for $p > 2$ its dimension is $qi - 1$, and for $p = 2$ it is $8j + k$ when $i = 4j + k$ for $1 \le k \le 3$ and $8j - 1$ when $i = 4j$.*

Proof. The Hurewicz map $\pi_*(V(0)) \to \pi_*(V(0) \wedge bo)$ is onto for $p = 2$, and for odd primes it is onto if we replace bo by $BP\langle 1\rangle$. Hence the stated result follows easily from 2.16 and 2.15 if we can show that the diagram in 2.15 commutes. For small values of i this can be shown by brute force computation using our knowledge of the relavant homotopy groups. The we can argue by induction on i as follows for p odd. In computing $R(p^i\iota)$ we produce a commutative diagram

$$\begin{array}{ccc} S^{-1} & \xrightarrow{h} & P_{-1-qi} \\ {\scriptstyle p^i\iota}\uparrow & & \uparrow{\scriptstyle i} \\ S^{-1} & \xrightarrow{v_1^i} & \Sigma^{-1-qi}V(0). \end{array}$$

(The bottom map must have Adams filtration i, and it is known that the only such map in this dimension is v_1^i.)

Now the degree p map on $V(0)$ is null homotopic, so the composite

$$S^{-1} \xrightarrow{p^{i+1}\iota} S^{-1} \xrightarrow{h} P_{-1-qi}$$

is also null and the dimension of $R'(p^{i+1})$ is at least $qi + q$. The composite

$$S^{-1} \xrightarrow{p^{i+1}\iota} S^{-1} \xrightarrow{h} P_{-1-qi-q}$$

factors through $\Sigma^{-1-qi-q}V(0)$ so the desired diagram commutes. For $p = 2$ we argue by induction on i four steps at a time. In computing $R(2^i\iota)$ we produce a commutative diagram

$$\begin{array}{ccc} S^{-1} & \xrightarrow{h} & P_{-1-2m} \\ {\scriptstyle 2^{i-3}\iota}\uparrow & & \uparrow{\scriptstyle i} \\ S^{-1} & \longrightarrow & P_{-1-2m}^{7-2m} \end{array}$$

for suitable m. It is known [DM] that the degree 16 map is null on P_{-1-2m}^{7-2m} for all m, so the composite

$$S^{-1} \xrightarrow{2^{i+1}\iota} S^{-1} \xrightarrow{h} P_{-1-2m}$$

is also null and the dimension of $R'(2^{i+1}\iota)$ is at least 8 more than that of $R'(2^{i-3}\iota)$ and again the desired diagram can be shown to commute. □

Now we will consider the spectrum J. For $p = 2$ it is the fibre of a certain map $bo \to \Sigma^4 bsp$ defined in terms of stable Adams operations. For $p > 2$ it is the fibre of a similar map $BP\langle 1\rangle \to \Sigma^q BP\langle 1\rangle$.

The following result is a straightforward calculation which is left to the reader. Closely related computations are done in several places, for example, [R1], [M4], and [DM].

Proposition 2.18. *For $p = 2$,*

$$\pi_i(\text{holim } J \wedge P_{-n}) = \begin{cases} \mathbf{Z}_2 & \text{if } i = -1 \text{ or } -2 \\ \mathbf{Z}/(2) & \text{if } i \equiv 0 \mod 4 \\ \mathbf{Z}/(2^k) & \text{if } i \equiv 2^{k+2} - 2 \mod 2^{k+3} \text{ for } k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

($\mathbf{Z}_p$ denotes the p–adic integers.) For $p > 2$ we have

$$\pi_i(\text{holim } J \wedge P_{-n}) = \begin{cases} \mathbf{Z}_p & \text{if } i = -1 \text{ or } -2 \\ \mathbf{Z}/(p^j) & \text{if } i = qsp^j - 2 \text{ for } s \text{ not divisible by } p \\ 0 & \text{otherwise.} \end{cases}$$

These computations also show the following.

Proposition 2.19. *The composite*

$$S^j \to S^{-1} \to \text{holim } (J \wedge P_{-n})$$

is essential only if $j = -1$ or $p = 2$ and the first map is η.

This says that the only elements in $\pi_*(J)$ which are root invariants (i.e., are contained in $R_J(\alpha)$ for some α) are the α_i described above and, for $p = 2$, ν which satisfies $\nu \in R(\eta)$. However, it does not say that these are the only cases where root invariants have nontrivial image under the J–theory Hurewicz map. To see this, consider the following diagram.

$$\begin{array}{ccccc} S^{k-1} & \xrightarrow{\beta} & S^{-n} & \longrightarrow & \Sigma^{-n} J \\ {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow \\ S^{-1} & \longrightarrow & P_{-n} & \longrightarrow & J \wedge P_{-n} \end{array}$$

Here β is an element in the coset $R(\alpha)$. The composite map to $\Sigma^{-n} J$ (the Hurewicz image of β) could be essential, while the composite map to $J \wedge P_{-n}$ (i.e., $R_J(\alpha)$) is null.

3. Alphas and betas

In 2.17 we saw that for p odd, the dimension of $R(p^i\iota)$ is $qi-1$ and the coset has Adams filtration i. Since there are no elements in that stem with higher Adams filtration, we can identify $R(p^i\iota)$ (up to unit scalar multiplication) with α_i, the element of order p in the image of the J–homomorphism, which is also the composite

$$S^{qi} \xrightarrow{i} \Sigma^{qi}V(0) \xrightarrow{\alpha^i} V(0) \xrightarrow{j} S^1 \tag{3.1}$$

where

$$\Sigma^q V(0) \xrightarrow{\alpha} V(0) \tag{3.2}$$

is a map due to Adams inducing an isomorphism in K–theory and multiplication by v_1 in BP–homology. For this reason we will frequently call this map v_1. This the first case of the *Greek letter construction*, which is described in more detail in Section 1.3 and 5.1 of [R1] and in [MRW].

The next Greek letter family (after the alphas) is constructed as follows. Let $V(1)$ denote the cofibre of the map α of (3.2). For $p \geq 5$ there is a map

$$\Sigma^{q(p+1)}V(1) \xrightarrow{\beta} V(1) \tag{3.3}$$

which induces multiplication by v_2 in BP–homology. We will frequently call this map v_2. The composite

$$S^{q(p+1)i} \xrightarrow{i} \Sigma^{q(p+1)i}V(1) \xrightarrow{\beta^i} V(1) \xrightarrow{j} S^{q+2} \tag{3.4}$$

is $\beta_i \in \pi_{q(p+1)i-q-2}(S^0)$.

In this section we will prove

Theorem 3.5. *For $p \geq 5$, $\beta_i \in R(\alpha_i)$ for all $i > 0$.*

There are difficulties with generalizing the Greek letter construction having to do with the fact that the desired maps do not always exist, e.g. α of (3.2) does not exist for $p = 2$ and β of (3.3) does not exist for $p = 3$. However one always has the root invariant, which suggests the following.

Definition 3.6. *The Greek letter elements are given by $\alpha_i = R(p^i\iota)$, $\beta_i = R(\alpha_i)$, etc.*

For $p \geq 3$ this α_i coincides with the usual α_i of (3.1) and for $p \geq 5$ this β_i coincides with the β_i of (3.4) by 3.5.

In general we expect that the root invariant converts a v_i–periodic family to a v_{i+1}–periodic family. This conjecture is formulated more precisely in [MR].

In order to prove 3.5 it is useful to have a secondary root invariant R'' for p odd which assigns to each element in $\pi_*(V(0))$ a coset in $\pi_*(V(1))$. Then we will show that $R''_{BP}(v_1^i) = -v_2^i$ and prove 3.5 using methods similar to the proof of 2.17.

Lemma 3.7. *(a) Let p be an odd prime and $n \geq m \geq 1$. Then the degree p map on the space P_{mq-1}^{nq+q} can be factored uniquely as*

$$P_{mq-1}^{nq+q} \xrightarrow{j} P_{mq+q-1}^{nq+q} \xrightarrow{\tilde{\alpha}} P_{mq-1}^{nq} \xrightarrow{i} P_{mq-1}^{nq+q}$$

where j and i are the evident pinch and inclusion maps.

Moreover the map $\tilde{\alpha}$ is a K–theoretic equivalence. We will denote its cofibre by $\bar{P}_{mq-1}^{nq}$; when $n = \infty$ we will denote it by $\bar{P}_{mq-1}$. In particular when $m = n$, $\tilde{\alpha}$ is the same as the Adams self–map

$$\alpha : \Sigma^{mq+q-1}V(0) \longrightarrow \Sigma^{mq-1}V(0)$$

and $\bar{P}_{mq-1}^{mq} = \Sigma^{mq-1}V(1)$.

(b) There are cofibre sequences

$$\Sigma^{mq-1}V(1) \xrightarrow{i} \bar{P}_{mq-1} \xrightarrow{j} \bar{P}_{mq+j-1}$$

and the inverse limit holim $\bar{P}_{qm-1}$ *is equivalent to $\Sigma^{-1}V(0)$. For $p > 3$ each $\bar{P}_{qm-1}$ is a module spectrum over $V(0)$.*

(c) For $p \geq 3$ the cofibres of the maps $\tilde{\alpha}j$, $i\tilde{\alpha}$ and $\tilde{\alpha}$ of (a) all admit self maps inducing multiplication by v_1 in BP–homology, and these are compatible with the one on $V(0) \wedge P_{mq-1}^{nq+q}$ induced by the Adams map on $V(0)$. For $p \geq 5$ these maps are uniquely determined by their compatibility.

(d) For $p \geq 5$ and $n \geq m$, the map $v_1 : \Sigma^q \bar{P}_{mq-1}^{q(n+p)} \to \bar{P}_{mq-1}^{q(n+p)}$ of (c) admits a unique factorization of the form

$$\Sigma^q \bar{P}_{mq-1}^{q(n+p)} \xrightarrow{j} \Sigma^q \bar{P}_{q(m+p)-1}^{q(n+p)} \xrightarrow{\tilde{\beta}} \bar{P}_{mq-1}^{nq} \xrightarrow{i} \bar{P}_{mq-1}^{q(n+p)}$$

where j and i are the evident pinch and inclusion maps. This can also be done when $n = \infty$, in which case the factorization through i is vacuous. Moreover the map $\tilde{\beta}$ is a $K(2)$–theoretic equivalence. When $m = n$, $\tilde{\beta}$ is the same as the Toda–Smith self map

$$\Sigma^{q(m+p+1)+1}V(1) \xrightarrow{\beta} \Sigma^{mq+1}V(1).$$

(e) $H^*(\bar{P}^{p(m+1)q}_{pmq-1})$ is a free module over $A(1)$ (the subalgebra of the Steenrod algebra A generated by $\mathcal{P}^1$ and the Bockstein β for p odd and by Sq^1 and Sq^2 for $p = 2$) on a single generator in dimension $pqm - 1$.

Most of (a) is proved in Proposition 7 of [G1].

For $p = 2$ the factorization exists only when m is even and n is odd and it is not unique. One of the choices does yield a K–theoretic equivalence. If we replace the degree 2 map by the one of degree 16, we get a unique factorization for all m and n of the form

$$P^{2n+8}_{2m-1} \longrightarrow P^{2n+8}_{2m+7} \longrightarrow P^{2n}_{2m-1} \longrightarrow P^{2n+8}_{2m-1}.$$

These maps are studied in [DM].

3.7 (b) enables us to define the secondary root invariant. This will be done below in 3.8. 3.7 (c), (d) and (e) are needed to prove 3.5.

Proof of 3.7. (a) The existence of the factorization on each end follows from the fact that the degree p maps on the mod p Moore spectra P^{mq}_{mq-1} and P^{nq+q}_{nq+q-1} are null homotopic. (This is not true when $p = 2$.) Uniqueness follows from the fact that the group $[\Sigma^{i+1}V(0), \Sigma^i V(0)]$ is trivial for the relevant i.

To show that the map is a K–theoretic equivalence it suffices to consider the case $m = n$, because if it is as claimed, then $\Sigma^2\bar{P}^{nq}_{mq-1}$ (for general m and n) has a filtration in which there are $n - m + 1$ subquotients, each of which is a suspension of $V(1)$ and therefore K–theoretically acyclic. For the case $m = n$ it is easy to check that $H^*(\bar{P}^{nq}_{mq-1})$ has the appropriate actions of the Steenrod operations Q_0 and Q_1, since it is a subquotient of $H^*(V(0) \wedge P^{mq+q}_{mq-1})$.

(b) From (a) we get a commutative diagram

$$\begin{array}{ccccccc}
P^{mq+q}_{mq-1} & \xrightarrow{j} & \Sigma^{mq+q-1}V(0) & \xrightarrow{\alpha} & \Sigma^{mq-1}V(0) & \xrightarrow{i} & P^{mq+q}_{mq-1} \\
{\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\
P_{mq-1} & \xrightarrow{j} & P_{mq+q-1} & \xrightarrow{\tilde{\alpha}} & P_{mq-1} & = & P_{mq-1} \\
{\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow \\
P_{mq+q-1} & \xrightarrow{j} & P_{mq+2q-1} & \xrightarrow{\tilde{\alpha}} & P_{mq+q-1} & = & P_{mq+q-1}
\end{array}$$

which yields

$$\begin{array}{ccccc}
\Sigma^{mq+q-1}V(0) & \xrightarrow{\alpha} & \Sigma^{mq-1}V(0) & \longrightarrow & \Sigma^{mq-1}V(1) \\
{\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\
P_{mq+q-1} & \xrightarrow{\tilde{\alpha}} & P_{mq-1} & \longrightarrow & \bar{P}_{mq-1} \\
{\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow \\
P_{mq+2q-1} & \xrightarrow{\tilde{\alpha}} & P_{mq+q-1} & \longrightarrow & \bar{P}_{mq+q-1}
\end{array}$$

This gives the dsired cofibre sequence. The value of the homotopy inverse limit follows from Lin's theorem and the commutativity of the diagram

$$\begin{array}{ccccc} V(0)\wedge P_{mq-1} & \longrightarrow & \bar{P}_{mq-1} & \longrightarrow & V(0)\wedge P_{mq+q-1} \\ {\scriptstyle V(0)\wedge j}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle V(0)\wedge j}\downarrow \\ V(0)\wedge P_{mq+q-1} & \longrightarrow & \bar{P}_{mq+q-1} & \longrightarrow & V(0)\wedge P_{mq+2q-1}. \end{array}$$

(c) We will construct the map on the cofibre of

$$P^{nq+q}_{mq+q-1} \xrightarrow{i\tilde{\alpha}} P^{nq+q}_{mq-1}$$

which we denote here by D^{nq}_{mq-1}. The other cases of the (c) are similar and are omitted. We have a commutative diagram

$$\begin{array}{ccccc} P^{nq+q}_{mq-1} & = & P^{nq+q}_{mq-1} & \longrightarrow & * \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle p} & & \downarrow \\ P^{nq+q}_{mq+q-1} & \xrightarrow{i\tilde{\alpha}} & P^{nq+q}_{mq-1} & \longrightarrow & D^{nq}_{mq-1} \\ \downarrow & & \downarrow{\scriptstyle i} & & \| \\ \Sigma^{mq}V(0) & \longrightarrow & P^{nq+q}_{mq-1}\wedge V(0) & \longrightarrow & D^{nq}_{mq-1} \end{array}$$

In order to get a self map on D^{nq}_{mq-1} it suffices to show that the following diagram commutes.

$$\begin{array}{ccc} \Sigma^{mq+q}V(0) & \longrightarrow & \Sigma^q P^{nq+q}_{mq-1}\wedge V(0) \\ \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_1} \\ \Sigma^{mq}V(0) & \longrightarrow & P^{nq+q}_{mq-1}\wedge V(0) \end{array}$$

To compare the two maps from $\Sigma^{mq+q}V(0)$ to $P^{nq+q}_{mq-1}\wedge V(0)$ we need to compute π_{mq+q} and π_{mq+q+1} of the target.

In this range of dimensions the target can be filtered into subquotients

$$\Sigma^{mq-1}V(0),\ \Sigma^{mq}V(0),\ \Sigma^{mq+q-1}V(0), \text{ and } \Sigma^{mq+q}V(0).$$

The relevant generators of $\pi_*(V(0))$ are $\iota\in\pi_0$, $\alpha_1\in\pi_{q-1}$ and $v_1\in\pi_q$. From this we see easily that $\pi_{mq+q+1}(P^{nq+q}_{mq-1}\wedge V(0))$ is trivial while $\pi_{mq+q}(P^{nq+q}_{mq-1}\wedge V(0))$ is generated by v_1 and possibly (when $m\equiv -1$ mod p) ι on the relavant subquotients. Both of these are detected by the BP–Hurewicz homomorphism. Therefore, since the diagram commutes in BP–homology, it commutes up to homotopy.

Hence we have a self map v_1 on D^{nq}_{mq-1}. It is unique if the group $[\Sigma^{mq+q+1}V(0), D^{nq}_{mq-1}]$ is trivial. A simple calculation (left to the reader) shows this to be true for $p\geq 5$.

(d) We will establish the factorization through j only, the argument for i being similar. We will use the method of elementary obstruction theory. We need to show that $\tilde{\beta}$ is null homotopic on $\Sigma^q \bar{P}_{mq-1}^{q(m+p-1)}$. It has nontrivial mod p homology in dimensions $q(m+p+i)+\epsilon$ for $0 \leq i \leq p-1$ and $\epsilon = -1, 0, q$ and $q+1$. In particular its homology is concentrated in dimensions congruent to $-1, 0$ and 1 mod q, so we need to study the homotopy of the target in those dimensions up to $q(pm+p+1)+1$.

The homotopy of $V(1)$ up to dimensions $2p^2$ (which is all we need her) can be computed by means of of the Adams spectral sequence, which has no differential or extension problems in this range. The E_2–term is spanned by the elements $\iota \in E_2^{0,0}$, $h_0 \in E_2^{1,q}$, $b_0 \in E_2^{2,pq}$, $h_1 \in E_2^{1,pq}$, $h_0 b_0 \in E_3^{2,(p+1)q}$, $g_0 \in E_2^{2,(p+2)q}$, and $v_2 \in E_2^{1,2p^2-1}$.

Hence, $\pi_*(V(1))$ is concentrated in dimensions congruent to $0, -1, -2$ and -3 mod q in the relevant range, i.e. up to $(p+1)q+2$. The only elements in dimensions divisible by q are $\iota \in \pi_0(V(1))$ and $v_2 \in \pi_{(p+1)q}(V(1))$. Since the target is built up out of $(qk-1)^{\text{th}}$ suspensions of $V(1)$, its homotopy is concentrated in dimensions congruent to $-1, -2, -3$ and -4 mod q. The first element in a dimension congruent to -1 mod q *not* detected by the mod p Hurewicz map is $v_2 \in \pi_{q(m+p+1)-1}$, i.e. v_2 on the first copy of $V(1)$.

The only overlap between the homology of the source and the homotopy of the target in the range of interest occurs in dimensions congruent to -1 mod q. (Note that since p is at least 5, q is at least 8. This part of the argument would break down for $p = 3$.) We do not have to worry about elements in the homotopy of the target that are detected by the mod p Hurewicz map because the map $\tilde{\beta}$ is trivial in mod p homology. The first possible obstruction is therefore $v_2 \in \pi_{q(m+p+1)-1}(\bar{P}_{mq-1})$, which lies in a dimension where the source has no homology.

This shows that the factorization through j exists. To establish its uniqueness we replace the homology of the source by that of its suspension. In this case the overlap with the homotopy of the target is trivial, so the factorization is unique.

The arguments for the existence and uniqueness of the factorization through i are similar and are omitted.

The element $v_2 \in \pi_{q(m+p+1)-1}(\bar{P}_{mq-1})$ could be and indeed is an obstruction to pinching the map further. In other words the map $\tilde{\beta}$ is essential on the bottom cell of it source. To see this note that in the case $m = n$, the cofibre of $\tilde{\beta}$ is a retract of $V(1) \wedge P_{mq-1}^{q(m+p+1)}$, from which it follows that the Steenrod operation Q_2 acts nontrivially on its cohomology. The cofibre is therefore a suspension of $V(2)$ and is K(2)–theoretically acyclic.

It follows that $\tilde{\beta}$ is a K(2)–equivalence as claimed.

(e) is a routine computation that is left to the reader. □

Definition 3.8. *For* $\alpha \in \pi_t(V(0))$*, the* **secondary root invariant** $R''(\alpha)$ *is the coset of* $\pi_*(V(1))$ *obtained as follows. Find the smallest* m *such that the composite*

$$S^{t-1} \xrightarrow{\alpha} \Sigma^{-1}V(0) \xrightarrow{h} \bar{P}_{-qm-1}$$

is essential and lift that composite to $\Sigma^{-qm-1}V(1)$*. For a ring spectrum* E*,* $R''_E(\alpha) \subset E_*(V(1))$ *is similarly defined.*

Lemma 3.9. *With notation as above, for* $p \geq 3$*,* $R''_{BP}(v_1^i) \ni -v_2^i$*.*

Proof. Recall that $BP_*(P_{qk-1})$ is generated as a BP_*–module by elements

$$b_m \in BP_{qm-1}(P_{qk-1}) \text{ for } m \geq k$$

with relations

$$\Sigma c_j b_{m+j} = 0 \text{ for each } m,$$

where the coefficients c_j are those of the p–series

$$[p](x) = \sum c_j x^{(p-1)j+1}$$

defined in terms of the formal group law. It follows that $BP_*(\bar{P}_{qk-1})$ has a similar description with the additional relation $pb_m = 0$.

The mod p reduction of the p–series has the form

$$v_1 x^p + v_2 x^{p^2} + \text{ higher terms.}$$

It follows that in $BP_*(\bar{P}_{-qk-1})$ we have

$$v_1^i b_0 = \begin{cases} 0 & \text{if } k < pi \\ -v_2^i b_{-pi} & \text{if } k = pi, \end{cases}$$

which gives the desired result. □

For $p \geq 5$, v_2^i can be regarded as an element in $\pi_*(V(1))$ using the map β of (3.3). In order to show that $R''(v_1^i) = -v_2^i$, we need the hypothesis of the analog of 2.15 for the secondary root invariant, namely

Lemma 3.10. *For $p \geq 5$ the following diagram commutes for all $t > 0$.*

$$\begin{array}{ccc} \Sigma^{-1}V(0) & \xrightarrow{h} & \bar{P}_{-ptq-1} \\ {\scriptstyle \alpha^t}\uparrow & & \uparrow{\scriptstyle i} \\ \Sigma^{tq-1}V(0) & \xrightarrow{-v_2^t} & \Sigma^{-ptq-1}V(1) \end{array}$$

where v_2^t denotes the composite

$$\Sigma^{|v_2^t|}V(0) \longrightarrow \Sigma^{|v_2^t|}V(1) \xrightarrow{\beta^t} V(1).$$

Proof. We will argue by induction on t, the result being trivial for $t = 0$. Consider the diagram

$$\begin{array}{ccccc} & & \Sigma^{-1}V(0) & \xrightarrow{h} & \bar{P}_{-ptq-1} \\ & & {\scriptstyle \alpha^t}\uparrow & & \uparrow{\scriptstyle i} \\ \Sigma^{tq+q-1}V(0) & \xrightarrow{\alpha} & \Sigma^{tq-1}V(0) & \xrightarrow{-v_2^t} & \Sigma^{-ptq-1}V(1) \end{array}$$

The composite $h\alpha^{t+1}$ is null homotopic (since $v_2\alpha$ is), so lifting h to $\bar{P}_{ptq-pq-1}$ gives a map

$$\Sigma^{tq+q-1}V(0) \xrightarrow{f_{t+1}} \bar{P}^{-ptq}_{-ptq-pq-1}.$$

This map has Adams filtration $t + 1$ since α and v_2 each have Adams filtration 1. The calculations in 3.9 show that this map is the canonical extension (using the $V(0)$–module structure on the target) of some representative of the Ext element $-v_2^{t+1}$ on the bottom cell, and is therefore essential.

We will show that this determines the map uniquely by showing that there are no elements in that dimension of higher Adams filtration. Here we will make use of 3.7(e), which says that $H*(\bar{P}^{-ptq}_{-ptq-pq-1})$ is a free module over $A(1)$ on a single generator. Thus its Adams E_2–term (up to suspension) is

$$\mathrm{Ext}_A(A(1), \mathbf{Z}/(p)).$$

We can get an upper bound on the size of this by looking at the E_1–term of the May spectral sequence, which is

$$\mathbf{Z}/(p)[v_n : n \geq 2] \otimes E(h_{i,j} : i \geq 1, j \geq 0, i + j \geq 2) \otimes \mathbf{Z}/(p)[b_{i,j} : i \geq 1, j \geq 0, i + j \geq 2]$$

where the bigradings of v_n, $h_{i,j}$ and $b_{i,j}$ are $(1, 2p^n - 1)$, $(1, 2p^j(p^i - 1))$ and $(2, 2p^{j+1}(p^i - 1))$ respectively. From this we see that there is a vanishing line of slope $1/(2p^2 - 2)$ (generated by powers of v_2), and the only generators lying above the line through the origin with that slope are $h_{1,1}$ and $h_{2,0}$.

It follows that there are no elements of higher Adams filtration, as claimed. Hence f_{t+1} is the composite

$$\Sigma^{tq+q-1}V(0) \xrightarrow{-v_2^{t+1}} \Sigma^{-ptq-pq-1}V(1) \xrightarrow{i} \bar{P}_{-ptq-pq-1}$$

and the result follows. □

We will need the following facts about the behavior of the image of J in the EHP sequence for odd primes. A proof can be found in [Th] or [G1].

Lemma 3.11. *Let $\bar{\alpha}_t$ and α_t denote the generator and an elkement of order p in the image of J in dimension $qt-1$. (When p does not divide t, the two elements are the same.) Then the following differentials occur in the EHP spectral sequence.*

$$d_{2j}(\phi_{2m}(\iota)) = \phi_{2(m-j)}(\bar{\alpha}_j) \qquad \textit{and} \qquad d_{2j+1}(\phi_{2(m-i)+1}(\alpha_i)) = \phi_{2(m-i-j)}(\bar{\alpha}_{j+i})$$

where $j = 1 + \upsilon_p(m)$. Moreover the element $\phi_{2(m-i)+1}(\alpha_i)$ for $i \leq 1 + \upsilon_p(m)$ is a permanent cycle corresponding to the element of order p^i in the image of J in the $(qm-1)$–stem.

We are now ready to prove part of 3.5. It follows from 3.9 and 3.10 that $-v_2^t \in R''(v_1^t)$. Hence we get a commutative diagram

$$\begin{array}{ccccc}
S^{tq-1} & \xrightarrow{v_1^t} & \Sigma^{-1}V(0) & \xrightarrow{j} & S^0 \\
{\scriptstyle -v_2^t}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
\Sigma^{-ptq-1}V(1) & \xrightarrow{i} & \bar{P}_{-ptq-1} & \xrightarrow{j} & \Sigma P_{-ptq-1+q} \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} & & \\
\Sigma^{-ptq+q}V(0) & \xrightarrow{i} & \Sigma P_{-ptq+q-1} & & \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} & & \\
S^{-ptq+q+1} & \xrightarrow{i} & \Sigma P_{-ptq+q} & &
\end{array} \tag{3.12}$$

The top composite is α_t and the left composite is $-\beta_t$ by definition. The conclusion of 3.5 follows provided that the indicated map $S^{tq-1} \to \Sigma P_{-ptq+q}$ is essential, i.e. if we can show that the dimension of $R(\alpha_t)$ is not greater than expected. We will do this with BP–theoretic methods. The indicated map $S^{tq-1} \to \Sigma P_{-ptq+q-1}$ has Adams–Novikov filtration 1. It is detected by a certain Ext element $\phi_{2(1-pt)}(\beta_t)$ which we will say more about below. Showing this element is nontrivial will prove that the map in question is essential. Composing with the pinch map to ΣP_{-ptq+q} is null only if the map lifts to S^{-ptq+q}. This lifting, if it exists, must also have Adams–Novikov filtration 1. It is known that the only such elements are in the image of the J–homomorphism, and their behavior in the EHP sequence is well enough understood (3.11) that we know that no such element could map to $\phi_{2(1-pt)-1}(\beta_t)$. Thus the proof of 3.5 will be complete if we can show that the Adams–Novikov Ext element $\phi_{2(1-pt)-1}(\beta_t)$ is nontrivial. Now we will define these elements precisely and show that they are nonzero. Recall ([R1]) that for a connective p–local spectrum X (such as P_{qm-1}) there is an Adams–Novikov spectral sequence converging to $\pi_*(X)$ with

$$E_2 = \mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(X)),$$

which we will abbreviate by $E_2(X;\, BP)$. A short exact sequence in the second variable induces a long exact sequence of Ext groups in the usual way. However a cofibre sequence of spectra need not induce a short exact sequence in BP–homology. For this reason it is more convenient to work with the spectra P_{qm-1} rather than P_{qm}. Consider the filtration of P_{qm-1} by even–dimensional skeleta,

$$P_{qm-1}^{qm} \subset P_{qm-1}^{qm+q} \subset \cdots P_{qm-1}.$$

It is easily shown that each inclusion induces a monomorphism in BP–homology, i.e. we have

$$BP_*(P_{qm-1}^{qm}) \subset BP_*(P_{qm-1}^{qm+q}) \subset \cdots BP_*(P_{qm-1}).$$

This filtration of $BP_*(P_{qm-1})$ leads in the usual way to a trigraded spectral sequence converging to

$$E_2(P_{qm-1};\, BP)$$

with

$$E_1^{s,t,u} = \begin{cases} E_2^{s,t}(P_{uq-1}^{uq};\, BP) & \text{for } u \geq m \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

and

$$d_r : E_r^{s,t,u} \longrightarrow E_r^{s+1,t,u-r}$$

We call this the *filtration spectral sequence*. Notice that each P_{uq-1}^{uq} is a suspension of $V(0)$. Given $x \in \mathrm{Ext}^{s,t}(V(0))$, we denote the corresponding element in $E_1^{s,t+uq-1,u}$ by

$$\lambda_{uq-1}x. \tag{3.14}$$

The element

$$\beta_t \in \mathrm{Ext}^{1,(p+1)tq-q}(V(0))$$

is defined as follows. It is known that

$$E_2^{0,*}(V(1);\, BP) = \mathbf{Z}/(p)[v_2]$$

with $v_2^t \in E_2^{0,(p+1)tq}$. The short exact sequence

$$0 \longrightarrow BP_*(\Sigma^q V(0)) \xrightarrow{v_1} BP_*(V(0)) \longrightarrow BP_*(V(1)) \longrightarrow 0$$

induces a connecting homomorphism

$$E_2^{s,t}(V(1);\, BP) \xrightarrow{\delta_1} E_2^{s+1,t-q}(V(0);\, BP)$$

and we define

$$\beta_t = \delta_1(v_2^t). \tag{3.15}$$

To finish proving 3.5 we need to show that the element $\lambda_{-ptq+q-1}\beta_t$ (defined by 3.14 and 3.15) is a nontrivial permanent cycle in the filtration spectral sequence 3.13. 3.10 implies that it is a permanent cycle both in the filtration spectral sequence and in the Adams–Novikov spectral sequence for $P_{-ptq+q-1}$. We only have to show that it is not the target of a differential in either spectral sequence. In the filtration spectral sequence the source of a differential hitting it must have Adams–Novikov filtration zero. We know that

$$E_2^{0,*}(V(0);\, BP) = \mathbf{Z}/(p)[v_1]$$

with $v_1^t \in E_2^{0,tq}$. Thus the source of the prospective differential must have the form $\lambda_{qm-1}v_1^n$ for suitable m and n. The behavior of these elements is known; each is either a permanent cycle or supports a differential of the form

$$d_r(\lambda_{qm-1}v_1^n) = \lambda_{q(m-r)-1}v_1^{n+r-1}h_0$$

for some r, where h_0 is the generator of $\mathrm{Ext}^{1,q}(V(0))$. (This can be seen by translating 3.11 to this setting. $\bar{\alpha}_j$ and $\phi_{2(m-i)+1}(\alpha_i)$ correspond to $v_1^{j-1}h_0$ and $\lambda_{q(m-i)-1}v_1^i$ respectively.) It follows that $\lambda_{-ptq-1}\beta_t$ is never the target of a differential in the filtration spectral sequence. In the Adams–Novikov spectral sequence it cannot be the target of a differential for sparseness reasons. This completes the proof of 3.5.

We do not know how this calculation goes at $p = 3$. For $p = 2$ it is discussed in [M1] and [M2].

It is instructive to use the same methods to compute $R(\beta_1)$. One would expect to get γ_1 as defined in [MRW], but this would contradict the odd primary analog of Jones' Theorem (2.2) since the dimension of γ_1, $(p^2-1)q-3$, is less than p times that of β_1, $pq-2$.

It turns out that γ_1 is in the indeterminacy, i.e., it maps to zero in the homotopy of $P_{-(p^2-p-1)q}$. This follows from the fact (proved in [MRW]) that $\gamma_1 = -\alpha_1\beta_{p-1}$ and the Steenrod operation $\mathcal{P}^1$ acts nontrivially on the bottom mod p cohomology class on $P_{-(p^2-p-1)q}$.

Further computations (which we leave to the interested reader) reveal that the algebraic root invariant of β_1 is actually $\beta_{p/p}$. This is the element supporting the Toda differential (the first nontrivial differential in the Adams–Novikov spectral sequence at an odd prime) $d_{q+1}(\beta_{p/p}) = \alpha_1\beta_1^p$. Thus Theorem 2.9 tells us that the homotopy root invariant lies in a lower stem. A little more work shows that it is β_1^p as Jones' Theorem would lead us to expect.

We will end this section by outlining a what may be a program for showing that $R(\beta_t) \ni \gamma_t$ for $t > 1$ and $p \geq 7$. Let $\bar{\bar{P}}_{pqm-1}$ be the cofibre of the map β of 3.7(d); it is built up out of suspensions of the 8–cell complex $V(2)$. Let $\bar{\bar{P}}_{p^2mq-1}^{p^2(m+1)q-q}$ be the fibre of the pinch map $\bar{\bar{P}}_{p^2mq-1} \to \bar{\bar{P}}_{p^2(m+1)q-1}$. Then presumably one has analogs of 3.7(d) and (e), namely $H^*(\bar{\bar{P}}_{p^2mq-1}^{p^2(m+1)q-q})$ is a free module over $A(2)$ on a single generator and for $p \geq 7$ there is a map

$$\Sigma^{(p+1)q}\bar{\bar{P}}_{p^2q(m+1)-1} \xrightarrow{\tilde{\gamma}} \bar{\bar{P}}_{p^2qm-1}$$

extending the classical map

$$\Sigma^{(p^2+p+1)q}V(2) \xrightarrow{\gamma} V(2)$$

The generalization of the secondary root invariant (3.8) is as follows. First, observe that we could have defined R'' to take values in $\pi_*(\bar{\bar{P}}_{p^2mq-1}^{p^2(m+1)q-q})$ for suitable m. (We suspect that for $p \geq 5$ the homotopy type of $\bar{P}_{p^2mq-1}^{p^2(m+1)q-q}$ is, up to suspension, independent of m, and similarly for $\bar{\bar{P}}_{p^2mq-1}^{p^2(m+1)q-q}$ for $p \geq 7$.) The *tertiary root invariant* R''' should be defined on $\pi_*(V(1))$ and take values in $\pi_*(\bar{\bar{P}}_{p^2mq-1}^{p^2(m+1)q-q})$. Then one can prove an analog of 3.10, i.e. that for $p \geq 7$, $R'''(v_2^t) = -v_3^t$. This leads us to a diagram similar to (3.12).

To complete the proof one needs to show that $\lambda_{-p^2tq-1}\gamma_t$ is nontrivial. It has Adams–Novikov filtration 2 and therefore could be the target (in the filtration spectral sequence) of a differential coming from Adams–Novikov filtration 1, e.g. from an element of the form $\lambda_{qm-1}\beta_n$. The behavior of these elements in the filtration spectral sequence is not yet known, so the program is incomplete. Indeed the discussion above indicates that when $t = 1$, $\lambda_{-p^2q-1}\gamma_1$ is in fact trivial.

4. Unstable homotopy

In this section we will look at root invariants and their implications in the actual EHP sequence. In the EHP sequence we are interested in those elements which are Hopf invariants of other elements. This could be a stable element, in which case we are interested in the element itself. Or, it could be an unstable element, in which case we are interested in the Whitehead product that it becomes. The first of these gives the progeny and the second gives the target set. We begin by defining the progeny of an element. Recall from 1.2 that the Hopf invariant, HI(α), is the coset in the E_1 term of the EHP spectral sequence which projects to the class in E_∞.

Definition 4.1. *The* **progeny** *of a stable element α is given by*

$$\mathrm{Prog}(\alpha) = \{\beta : \alpha \in HI(\beta)\}.$$

In other words we ask for the set of β whose name in $E_\infty^{s,t}$ is $\phi_s(\alpha)$ (see 1.11). Hence if β' differs from β by an element which desuspends further, then it is in Prog(α) if β is. When describing Prog(α) we will only list one element from each of these cosets. The same is true of the set Targ(α), to be defined below (4.3).

For example we have for $p = 2$,

$$\mathrm{Prog}(\iota) = \{\eta,\, \nu,\, \sigma\},$$

the elements with Hopf invariant one;

$$\mathrm{Prog}(\eta) = \{\eta^2,\, 2\nu,\, 2\sigma\};$$
$$\mathrm{Prog}(\nu) = \{\nu^2,\, \eta_j \text{ for } j \geq 3\}\text{(see [M3]) and}$$
$$\mathrm{Prog}(\sigma) = \{\sigma^2,\, \nu^*\},$$

where ν^*, Toda's notation, is the element corresponding to h_2h_4. Details on these calculations can be found in [M6] and [M7].

Notice that ν is unique among elements of Hopf invariant one in that its progeny is infinite.

For $p \geq 3$ we have

$$\mathrm{Prog}(\iota) = \{\alpha_1\} \text{ and}$$
$$\mathrm{Prog}(\alpha_1) = \{\alpha_2,\, \alpha_{3/2} \text{ (for } p = 3),\, \beta_1\}.$$

For $p \geq 5$, we conjecture that

$$\mathrm{Prog}(\beta_1) = \{\alpha_1\beta_1,\, \beta_2,\, \beta_{p^j/p^j-1} \text{ for } j \geq 1\}$$
$$\mathrm{Prog}(\beta_2) = \{\beta_3,\, \alpha_1\beta_{p^j/p^j},\, \beta_{p^j/p^j-2} \text{ for } j \geq 1\}.$$

Here of course we are using the notation of [MRW].

Proposition 4.2. *If* Prog(α) *is an infinite set then* α *is anomalous and therefore (1.12) a root invariant.*

Clearly the converse is not true since η is a root invariant. We are now prepared to define Targ. Recall that if α is a stable homotopy class, then $\phi_s(\alpha)$ is defined for s large enough.

Definition 4.3. *The* **target set** *of a stable element* α *is given by*

$$\mathrm{Targ}(\alpha) = \{\beta :\ d_r(\phi_s(\alpha)) = \phi_{s-r}(\beta) \qquad \textit{for some } s \textit{ and some } r > 1\}$$

in the stable EHP spectral sequence.

By requiring $r > 1$ in this definition, we are excluding (as uninteresting) the element $p\alpha$, since one always has

$$d_1(\phi_{2n+1}(\alpha)) = \phi_{2n}(p\alpha)$$
$$d_1(\phi_{2n}(\alpha)) = 0.$$

The importance of this set is that it is equivalent to the set of names that the Whitehead product of ι and α (at least when α is a double suspension) receive in the metastable range of the EHP spectral sequence. Determining this set is usually a very hard problem. For example the solution to the vector field problem ([T2], [A1]) gives

$$\mathrm{Targ}(\iota) = \{\omega_j\},$$

where ω_j is the jth generator of the image of the J homomorphism. The analogous statement for odd primes is (using the notation of 3.11)

$$\mathrm{Targ}(\iota) = \{\bar{\alpha}_j\}.$$

This is related to a result sometimes called the odd primary vector field theorem ([Th], [T2]).

This is true for all primes.

Proposition 2.18 suggests the following.

Conjecture 4.4. *For $p = 2$, if θ_j exists and has order two, then for large j,*

$$\mathrm{Targ}(\omega_j) = \{\theta_j, \omega_{j+1}\}.$$

The following is clear from the definitions.

Proposition 4.5. *An element is anomalous if its target set is infinite.*

Note that all v_1–periodic elements in SE_2 (as defined, for example, in [M4]) are in the union of the sets $\mathrm{Targ}(p^i)$.

Consider $\mathrm{Targ}(\nu)$. Theorem 1.12 implies that ν is anomalous since it is the root invariant of η, so its target set is infinite. Detailed calculations seem to give the following. It is rather easy to get the algebraic version.

Conjecture 4.6.

$$\mathrm{Targ}(\nu) = \{\nu^2, \sigma^2, \nu^*, x_{1+2^{1+j}} \text{ for } j \geq 4\},$$

where ν^* is as in $\mathrm{Prog}(\nu)$ *above and the $x_{1+2^{1+j}}$ are the elements corresponding to $h_2h_j^2$ recently discovered by Bruner* [B].

Notice that the stems of elements in $\mathrm{Targ}(\iota)$ grow linearly while those in $\mathrm{Targ}(\nu)$ grow exponentially.

More explicitly, 4.6 says that the behavior of ν in the stable EHP spectral sequence is as follows.

$$\begin{aligned}
d_1(\phi_{2n+1}(\nu)) &= \phi_{2n}(2\nu) \\
&\qquad \text{(we have decided to exclude this target in Targ)} \\
\phi_{8n}(\nu) &= d_4(\phi_{8n+4}(\iota)) \\
d_4(\phi_{8n+2}(\nu)) &= \phi_{8n-2}(\nu^2) \\
d_4(\phi_{8n+4}(\nu)) &= \phi_{8n}(\nu^2) \\
d_{12}(\phi_{16n+6}(\nu)) &= \phi_{16n-6}\sigma^2 \\
d_{16}(\phi_{32n+14}(\nu)) &= \phi_{32n-2}(\nu^*) \\
d_{2^j-1}(\phi_{k2^j-2}(\nu)) &= \phi_{(k-1)2^j-1}(x_{1+2^j}) \\
&\qquad \text{for } j \geq 5 \text{ and } k \text{ odd.}
\end{aligned}$$

The fact that $\nu \in R(\eta)$ means that $\phi_{-2}(\nu)$ is a nontrivial permanent cycle in the stable EHP spectral sequence. Notice that $\phi_n(\nu)$ lives longer as n approaches -3 2–adically.

Here is an odd primary example of the same phenomenon. The following differentials are also conjectured to occur in the stable EHP spectral sequence.

$$\begin{aligned}
d_1(\phi_{2i+1}\iota) &= \phi_{2i}(p\iota) \\
d_{2j+2}(\phi_{2sp^j}) &= \phi_{2(sp^j-2j-2)}(\bar{\alpha}_{j+1}) \\
&\qquad \text{for } s \not\equiv 0 \mod p \text{ and } j \geq 0 \\
d_{2j+3}(\phi_{2sp^j-1}\alpha_1) &= \phi_{2(sp^j-2j-4)}(\bar{\alpha}_{j+2}) \\
&\qquad \text{for } s \not\equiv 0 \mod p \text{ and } j \geq 0 \\
d_{2p-2}(\phi_{2pi-2}(\alpha_1)) &= \phi_{2pi-2p}(\beta_1) \\
d_2(\phi_{2s+3}(\beta_1)) &= \phi_{2s+1}(\alpha_1\beta_1) \text{ for } s \not\equiv 0 \mod p \\
d_2(\phi_{2s}(\beta_1)) &= \phi_{2s-2}(\alpha_1\beta_1) \text{ for } s \not\equiv 0 \mod p \\
d_{2p+3}(\phi_{2ps-2p+3}(\beta_1)) &= \phi_{2s-4p}(\beta_2) \text{ for } s \not\equiv 0 \mod p \\
d_{2p^{j+1}-2p+1}(\phi_{2p^{j+1}s-2p+3}(\beta_1)) &= \phi_{2p^{j+1}(s-1)}(\beta_{p^j/p^j-1}) \\
&\qquad \text{for } s \not\equiv 0 \mod p \text{ and } j \geq 1
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathrm{Targ}(\alpha_1) &= \{\beta_1,\, \bar{\alpha}_j \text{ for } j \geq 2\} \\
\mathrm{Targ}(\beta_1) &= \{\alpha_1\beta_1,\, \beta_2,\, \beta_{p^j/p^j-1} \text{ for } j \geq 1\}
\end{aligned}$$

and the target sets for ι and α_1 are linearly distributed while that of β_1 is exponentially distributed.

Again, since $\beta_1 \in R(\alpha_1)$, $\phi_{3-2p}(\beta_1)$ survives to LE_∞, and $\phi_n(\beta_1)$ lives longer as n approaches $2p-3$ 2–adically.

Proposition 4.7. *Suppose α is anomalous and is the root invariant for only one element. Let N_i denote the set of integers n such that $\phi_n(\alpha)$ projects to a nonzero cycle in LE_i. The N_i have the following properties.*

(1) $N_1 = \mathbf{Z}$.
(2) *The collection $\{N_i\}$ is a 2–adic neighborhood system for the integer n such that $\phi_n(\alpha)$ is a nonzero cycle in LE_∞.*
(3) *If n is in N_i, then so is $n + f(i)$, where $f(i)$ is as in 1.5.*

Proposition 4.10. *If $\alpha = R(\beta)$ and β corresponds to a class b in Ext such that bh_j is a nontrivial permanent cycle for all large j, then* $\mathrm{RTarg}(\alpha)$ *is exponentially distributed.*

Proof. Let n and m be the dimensions of α and β respectively. The discussion connected with 1.9 shows that multiplication by h_j in Ext is a monomorphism in a certain range that increases with j. We also note there that $HI(bh_j) = R(b)$. This implies that $bx_s \in \mathrm{Ext}(P_t)$ is a permanent cycle for $s = k2^{j-1}$ and $t = (k-1)2^j$.

Consider the diagram

$$\begin{array}{ccccc} P_t & \longrightarrow & P_{s-n} & \longrightarrow & P_s \\ & & {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle i} \\ & & S^{s+m} & \xrightarrow{\beta} & S^s \end{array}$$

where n is the smallest integer so that the indicated map from S^{s+m} to P_{s-n} is essential. The left vertical map is α on the bottom cell, and the right vertical map is the standard inclusion.

The map α lifts back to P_t and possibly to a stunted projective space with a still lower bottom dimensional cell, but not back to $P_{-\infty}$. The obstruction to lifting it back further is β_j. The diagram shows that its stem exceeds that of α by at least 2^j, which means that $\mathrm{RTarg}(\alpha)$ is exponentially distributed as claimed.

□

5. The proof of 1.12

Recall Theorem 1.12 asserts that a stable homotopy element is anomalous (1.11) iff it is a root invariant (1.10). As remarked above, it is easy to see that each root invariant is anomalous, so the object of this section is to prove the converse.

We begin by proving an algebraic analog of this statement. Recall [C] that for $p = 2$ there is an unstable Adams spectral sequence converging to $\pi_*(S^n)$ for each positive n. Its E_2–term is an *Ext* defined in a suitable category of unstable A–modules. For odd primes see [BC].

More explicitly one has a differential graded associative algebra Λ whose cohomology is the stable Adams E_2–term for the sphere. Λ is generated by elements λ_i of dimension i and cohomological degree 1. These are subject to certain relations similar to the Adem relations. This leads to an additive basis for Λ consisting of monomials

$$\lambda_{i_1}\lambda_{i_2}\lambda_{i_n}\cdots$$

satisfying

$$i_2 \leq 2i_1,\ i_3 \leq 2i_2, \cdots$$

For each positive n there is a subcomplex $\Lambda(n)$ spanned by monomials as above with $i_1 < n$. The cohomology of this complex is the unstable Adams E_2–term for S^n. There are short exact sequences of complexes

$$0 \to \Lambda(n) \to \Lambda(n+1) \to \Sigma^n\Lambda(2n+1) \to 0 \tag{5.1}$$

which correspond to the fibrations

$$\Omega^n S^n \to \Omega^{n+1}S^{n+1} \to \Omega^{n+1}S^{2n+1}.$$

The sequence 5.1 leads to a long exact sequence in cohomology which is the E_2–term analog of the EHP sequence. As before this leads one to an algebraic EHP spectral sequence as follows.

Proposition 5.2. *There is a spectral sequence converging to*

$$\mathrm{Ext}_A^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

with

$$E_1^{s,t,n} = E_2^{s-1,t-n}(S^{2n-1}) \text{ and } d_r : E_r^{s,t,n} \to E_r^{s+1,t,n-r}.$$

Our indexing for the unstable Adams spectral sequence here is such that $E_2^{s,t}(S^n)$ converges to a subquotient of $\pi_{n+t-s}(S^n)$.

A similar EHP property for the unstable Adams–Novikov spectral sequence is established in [BCR].

The spectral sequence of 5.2 has James periodicity comparable to that in 1.5.

Proposition 5.3. *For $r < j$,*

$$E_r^{s,t,n} = E_r^{s,t',n'}$$

for $t' - t = n' - n \equiv 0 \mod 2^j$.

As in 1.6 and 1.8 there are spectral sequences

$$\{SE_r^{s,t,n}\} \text{ and } \{LE_r^{s,t,n}\}.$$

In particular we have

$$LE_1^{s,t,n} = \mathrm{Ext}_A^{s-1,t-n}(\mathbf{Z}/(2), \mathbf{Z}/(2)).$$

The algebraic form of Lin's theorem says that these coincide and converge to $E_2(S^{-1})$.

In order to proceed further we must recall some of the calculations in [LDMA]. Let $A(i)$ be the subalgebra of the Steenrod algebra A generated by $\mathrm{Sq}^k : k \leq 2^i\}$. For any A–module M one has

$$\mathrm{Ext}_A(M, \mathbf{Z}/(2)) = \varprojlim \mathrm{Ext}_{A(i)}(M, \mathbf{Z}/(2)). \tag{5.4}$$

We will abbreviate $\mathrm{Ext}_{A(i)}(H^*(X), \mathbf{Z}/(2))$ by $\mathrm{Ext}_{A(i)}(X)$, and $\lim_{\leftarrow} Ext_{A(i)}(P_{-k})$ by $\mathrm{Ext}_{A(i)}(P_{-\infty})$.

In [LDMA] (Lemma 1.3) it is shown that

Proposition 5.5.

$$\mathrm{Ext}_{A(i+1)}(P_{-\infty}) = \sum_{k \in \mathbf{Z}} \mathrm{Ext}_{A(i)}(S^{2^{i+2}k-1}).$$

The map

$$\mathrm{Ext}_{A(i+1)}(P_{-\infty}) \to \mathrm{Ext}_{A(i)}(P_{-\infty})$$

sends the k^{th} copy of $\mathrm{Ext}_{A(i)}$ to the $(2k)^{th}$ copy of $\mathrm{Ext}_{A(i-1)}$, and the $(2k+1)^{th}$ copy of it is not in the image.

The group $\mathrm{Ext}_{A(i+1)}(P_{-\infty})$ can be calculated with a spectral sequence based on the skeletal filtration of $P_{-\infty}$. In this way we can assign to each element in this group a name consisting of a cell in dimension $2^{i+2}k - 1$ and an element in $\mathrm{Ext}_{A(i)}$. This can be regarded as a root invariant that assigns to each element in $\mathrm{Ext}_{A(i)}$ a coset in $\mathrm{Ext}_{A(i+1)}$. This idea is explored further by Shick [Sh].

Lemma 5.6. *In the $A(i)$–analog of the stable algebraic EHP spectral sequence, we denote the analog of*

$$\{LE_r^{s,t,n}\},$$

by

$$\{LE(i)_r^{s,t,n}\}.$$

Then for each i and s, there is an r such that for all t and n

$$LE(i)_r^{s,t,n} = LE(i)_\infty^{s,t,n}.$$

Proof. It is known that

$$\mathrm{Ext}_{A(i)}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = 0 \text{ for } (2^{i+1}-1)s < t,$$

i.e., it vanishes below a line of slope $1/(2^{i+1}-2)$. (One might call this a vanishing ledge rather than a vanishing line.) This can be seen by looking at the E_1–term of the appropriate May spectral sequence (see 3.2 in [R1]). It follows that

$$LE(i)_1^{s,t,n} = 0 \quad \text{for } (2^{i+1}-1)s < t-n+\epsilon$$

for a suitable constant ϵ.

It is also known that above a certain line of slope $1/2$, all elements of this Ext group are divisible by $h_0 \in \mathrm{Ext}^{1,1}$. Moreover, the first differential in this spectral sequence,

$$LE(i)_1^{s,t,2m+1} = \mathrm{Ext}_{A(i)}^{s-1,t-1-2m} \xrightarrow{d_1} LE(i)_1^{s+1,t,2m} = \mathrm{Ext}_{A(i)}^{s,t-2m},$$

is multiplication by h_0. This means that

$$LE(i)_2^{s,t,n} = 0 \quad \text{for } s > t-n+\epsilon'$$

for a some constant ϵ'.

Combining these two, we see that for $r \geq 2$, $LE(i)_r^{s,t,n}$ is nontrivial only when

$$\frac{t-n+\epsilon}{2^{i+1}-1} \leq s \leq t-n+\epsilon'.$$

It follows that the group $LE(i)_r^{s+1,t,n-r}$ (the target of a differential from $LE(i)_r^{s,t,n}$) is nontrivial only when

$$\frac{t-n+r+\epsilon}{2^{i+1}-1} \leq s+1 \leq t-n+r+\epsilon',$$

i.e., for only finitely many values of r.

Similarly, the group $LE(i)_r^{s-1,t,n+r}$, which would support a d_r whose target is in $LE(i)_r^{s,t,n}$, is nontrivial only when

$$\frac{t-n-r+\epsilon}{2^{i+1}-1} \leq s-1 \leq t-n-r+\epsilon',$$

i.e., for only finitely many values of r.

The result follows. □

It follows that if an element is anomalous over $A(i)$ then it occurs infinitely often in the E_∞–term and is a root invariant by 5.5. The following result shows that an anomalous element projects to a root invariant over each $A(i)$ for i sufficiently large.

Lemma 5.7. *If*

$$a \in \mathrm{Ext}_A(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

is anomalous then so is its projection

$$a(i) \in \mathrm{Ext}_{A(i)}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

for i sufficiently large.

Proof. Suppose a has bigrading (s,t). We can choose i so that for each n, $\mathrm{Ext}_{A(i)}(P_{-n})$ and $\mathrm{Ext}_A(P_{-n})$ are the same in a range which includes the element $x_n = \phi_n(a)$. By 5.6 we can choose r such that $LE(i)_r^{s,t+n,n} = LE(i)_\infty^{s,t+n,n}$. Since a is anomalous, there are infinitely many n for which x_n is a nontrivial r–cycle. By naturality its image $x_n(i)$ is also an r–cycle. If it is a boundary then so is its image in the spectral sequence which calculates $\mathrm{Ext}_{A(i)}(P_n)$. In this range this calculation is the same as that of $\mathrm{Ext}_A(P_n)$ where x_n has a nontrivial image in E_r. Therefore $x_n(i)$ survives to E_r and therefore (by 5.6) to E_∞. Hence $a(i)$ is anomalous as required. □

The following is a restatement of a result in [LDMA].

Lemma 5.8. *If $a(i)$ is as above, then it is equal to $\phi_{k(i)}(y(i))$, where $k(i) = m2^i - 1$ for some m, and $y_{(i)}$ is the image of some y which is in $\mathrm{Ext}_A(\mathbf{Z}/(2), \mathbf{Z}/(2))$. Moreover, $m = 0$ for i sufficiently large.*

It follows that a is the root invariant of y, which proves the Ext analog of 1.12. For 1.12 itself we need

Lemma 5.9. *Let x be an anomalous element. For each n we can compose x with the inclusion of the bottom cell in P_n and get an element in $\pi_*(P_n)$. Let s_n denote its Adams filtration. Then the set of such integers is bounded.*

Proof. This follows from the fact that $E_2(P_n)$ has a vanishing line in dimensions greater than n (and in all dimensions if n is even). The dimension of the anomalous element x must be positive. □

Now we are ready to finish the proof of 1.12. We will show that if x is anomalous, it must be a root invariant.

Suppose n is an integer mod $f(r)$ (with $f(r)$ as in 1.5) so that $\phi_n(x)$ represents a nonzero r–cycle. Thus the corresponding element in $\pi_*(P_n)$ lifts to $\pi_*(P_{n-r})$. The Adams filtration may drop with this lifting. Thus for each r and for each congruence class of n mod $f(r)$ such that $\phi_n(x)$ is an r–cycle, we get a pair (a,j) such that $\phi_{n-j}(a)$ is the Adams spectral sequence name of the lift of $\phi_n(x)$ in $\pi_*(P_{n-r})$. In particular, $\phi_{n-j}(a)$ is a nontrivial permanent cycle in the Adams spectral sequence for $\pi_*(P_{n-r})$.

If x is anomalous we can do this for each r. Since the Adams filtration of the lift can drop only finitely many times as r increases, the maximum j and the corresponding a are independent of r for r sufficiently large. Thus for each large r, there is a conguence class of $n \mod f(r)$ and a pair (a, j) such that $\phi_{n-j}(a)$ is the Adams spectral sequence name of a lifting of $\phi_n(x)$ to $\pi_*(P_{n-r})$.

This means that there is an integer k such that $\phi_k(a)$ is a nontrivial r-cycle for each r, and k is congruent to $n - j$ mod $f(r)$ for all r. Hence we can set $n = k + j$ and we have a diagram

$$\begin{array}{ccc} S^n & \longrightarrow & P_n \\ {\scriptstyle x}\big\uparrow & & \big\uparrow \\ S^{n+|x|} & \xrightarrow{\{a\}} & P_{n-j} \end{array}$$

where the bottom map represents $\phi_k(a)$. By hypothesis this map lifts to P_ℓ for all $\ell < n - j$. Therefore it gives us an element $y \in \pi_{n+|x|}(S^{-1})$, and $x \in R(y)$ by construction. This completes the proof of 1.12.

6. Improved James periodicity and related results

James periodicity (1.5) gives isomorphisms in the EHP spectral sequence

$$E_r^{k,n} \cong E_r^{k+(p-1)f(r),n+f(r)}.$$

The function $f(r)$ grows exponentially with r. In practice one finds that many differentials have James periods far shorter than guaranteed by 1.5. In this section we will indicate a program for getting shorter periods for some differentials. We will not give specific results as they would depend on detailed knowledge of the homotopy of certain cobordism spectra.

Recall that the EHP spectral sequence maps to the stable EHP spectral sequence, which is a reindexed form of the Atiyah–Hirzebruch spectral sequence for the stable homotpy of RP^∞. Differentials have the form

$$d_r(\phi_n(\alpha)) = \phi_{n-r}(\beta). \tag{6.1}$$

Given a homology theory E_* with a unit map $S^0 \to E$, we get a map from the stable EHP spectral sequence to the Atiyah–Hirzebruch spectral sequence for $E_*(RP^\infty)$. If α and β both have nontrivial images in $\pi_*(E)$, then the differential 6.1 could map to a differential in the spectral sequence for $E_*(RP^\infty)$. If this spectral sequence can be shown to have shorter James periodicity, then we may be able to get a shorter James period for the differential 6.1.

The spectra E that we will use are Thom spectra associated with connective covers of BO. Define an arithmetic function ϕ by

$$\phi(k) = \begin{cases} 2k & \text{if } k \equiv 1 \text{ or } 2 \mod 4 \\ 2k+1 & \text{if } k \equiv 0 \mod 4 \\ 2k+2 & \text{if } k \equiv 3 \mod 4. \end{cases}$$

Let $MO\langle\phi(k)\rangle$ denote the Thom spectrum associated with the $(\phi(k)-1)$–connected cover of BO. The function ϕ is chosen so that its image is all the dimensions in which $\pi_*(BO)$ is nontrivial. This is all for the prime 2. We leave the necessary modifications for odd primes to the interested reader.

The following result was first proved in [DGIM]; a more accessible proof is given in [MRay].

Theorem 6.2. *The homotopy type of $P_n \wedge MO\langle\phi(k)\rangle$ depends (up to suspension) only on the congruence class of n modulo 2^k.*

Corollary 6.3. *In the (reindexed) Atiyah–Hirzebruch spectral sequence use to calculate $MO\langle\phi(k)\rangle_*(RP^\infty)$, there are isomorphisms*

$$E_r^{s,t} = E_r^{s+2^k,t+2^k}.$$

Notice that this James periodicity is independent of r.

The first interesting case of this is $k = 2$. Here the Thom spectrum is $MSpin$, and the image of $\pi_*(S^0)$ is known to be $\mathbf{Z}/(2)$ in every dimension congruent to 1 or 2 mod 8. (The elements so detected are the μ's.) Thus any EHP differential whose target is one of these elements will be detected by this spectral sequence and have James period 4. The precise statement of a theorem about this would have to include a hypothesis to the effect that neither the source do target of the James periodic image of the differential be killed by an earlier differential not detected by $MSpin$.

We close with an amusing construction of some spectra paramtrized by the 2–adic integers $\mathbf{Z}_2$. Recall that

$$KO^*(RP^\infty) = \mathbf{Z}_2,$$

topologically generated by the class of the reduced canonical line bundle. For each 2–adic integer u , there is a Thom spectrum T_u with

$$T_n = \Sigma^{-n} P_n$$

when n is an ordinary integer. For each u there is a map

$$\Sigma^{-1} T_{u-1} \to T_u$$

and these can be used to form an inverse system and a spectrum X_u analogous to $P_{-\infty}$. Thus we have

$$X_0 = P_{-\infty} \qquad \text{and}$$
$$X_u = \Sigma^n X_{u-n}$$

for each 2–adic integer u and ordinary integer n.

The argument of [MRay] can be adjusted to prove an analog of 6.2, namely

$$T_u \wedge MO\langle\phi(k)\rangle \simeq T_{u+2^k} \wedge MO\langle\phi(k)\rangle.$$

For each X_u there is an analog of the stable EHP spectral sequence. For each r its E_r–term is some suspension of that of the usual stable EHP spectral sequence, since E_r depends only on the congruence class of u modulo a suitable power of 2. We can define regular and anomalous elements just as before, but if u is not an ordinary integer, there will be no root invariants. In fact the methods of [LDMA] will show the following.

Proposition 6.4.

$$\mathrm{Ext}_{A(i)}(X_u) = \oplus\Sigma^m \mathrm{Ext}_{A(i-1)}(\mathbf{Z}/(2)),$$

where the sum is over all m which are congruent to $u-1$ modulo 2^{i+1}.

Thus if u is not an ordinary integer, there is no m that is congruent to $u-1$ modulo all powers of 2 , so the inverse limit of these Ext groups, i.e., $\mathrm{Ext}_A(X_u)$, is trivial and X_u is contractible.

This construction also generalizes to odd primes in an obvious way with RP^∞ replaced by $B\Sigma_p$, and the real line bundle σ replaced by the complex reduced regular representation bundle. The $p-1$ stable summands of $B\mathbf{Z}/(p)$ are equivalent (after pinching out the bottom cell) to suspensions of the spectra $T_{i/(p-1)}$ for $1 \le i \le p-1$. In these cases the spectra $X_{i/(p-1)}$ for $1 \le i < p-1$ were shown to be contractible in [AGM].

References

[A1] J.F. Adams, Vector fields on spheres, *Ann. of Math.* **75**(1962), 603–632.

[A2] J.F. Adams, On the groups $J(X), IV$, *Topology* **5**(l966), 21–71.

[AGM] J.F. Adams, J.H.C. Gunawardena and H.R. Miller, The Segal conjecture for elementary abelian groups, *Topology* **24** (1985), 435–460.

[B] R. Bruner, A new differential in the Adams spectral sequence, *Topology* **23**(l984), 271–276.

[B6] A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector and J.W. Schlesinger, The mod p lower central series and the Adams spectral sequence, *Topology* **5**(1966), 331–342.

[BC] A.K. Bousfield, and E.B. Curtis, A spectral sequence for the homotopy of nice spaces, *Trans. Amer. Math. Soc.* **151**(1970), 457–479.

[BCR] M. Bendersky, E.B. Curtis and D.C. Ravenel, EHP sequences in BP theory, *Topology* **21**(l982), 373–391.

[BJM] M.G. Barrett, J.D.S. Jones and M.E. Mahowald, The Kervaire invariant problem, *Proceedings of the Northwestern University Homotopy Theory Conference, Contemp. Math. AMS* **19**(1983), 9–23.

[DGIM] D.M. Davis, S. Gitler, W. Iberkleid, and M.E. Mahowald, The orientability of vector bundles with respect to certain spectra, Boll. Soc. Mat. Mex. 26(1981), 49–55.

[DHS] E. Devinatz, M. Hopkins, and J. Smith, Nilpotence and stable homotopy, *Ann. of Math.* **128**(198), 207–242.

[DM] D.M. Davis and, M.E. Mahowald, Homotopy groups and some mapping telesopes, in *Algebraic Topology and Algebraic K–Theory*, Ann. Math. Studies **113**, 1987, pp. 126–151.

[G1] B. Gray, Unstable families related to the image of J, *Math. Proc. Camb. Phil. Soc,* **96**(1984), 95–113.

[G2] B. Gray, On the iterated suspension, *Topology,* **27**(1988), 301–310.

[J1] I.M. James, On the suspension sequence, *Ann. of Math.* **65**(1957), 74–107.

[J2] I.M. James, Spaces associated with Stiefel manifolds, *Proc. London Math. Soc.* **9**(1959), 115–140.

[Jo] J.D.S. Jones, Root invariants, cup–r products and the Kahn–Priddy theorem, *Bull. London Math. Soc.* **17**(1985), 479–483.

[KP] D.S. Kahn and S.B. Priddy, The transfer and stable homotopy, *Math. Proc. Camb. Phil. Soc.* **83**(1978), 103–111.

[L1] W.H. Lin, On conjectures of Mahowald, Segal and Sullivan, Math. Proc. Camb. Phil. Soc. **87**(1980), 449–458.

[L2] W.H. Lin, Algebraic Kahn–Priddy theorem, *Pacific J. Math.* **96**(1981), 435–455.

[LDMA] W.H. Lin, D.M. Davis, M.E. Mahowald and J.F. Adams, Calculations of Lin's Ext groups, *Math. Proc. Camb. Soc.* **87**(1980), 459–469.

[M1] M.E. Mahowald, Some homotopy classes generated by η_j, Algebraic Topology Aarhus 1978. Springer Lecture Notes in Math. 763(1979), 23-37.

[M2] M.E. Mahowald, The primary v_2–periodic family, *Math. Zeit.* **177**(1981), 381–393.

[M3] M.E. Mahowald, A new infinite family in ${}_2\pi_*^S$, *Topology* **16**(1977), 249–256.

[M4] M.E. Mahowald, The image of J in the EHP sequence, *Ann. of Math.* **116**(1982), 65–112.

[M5] M.E. Mahowald, Lin's theorem and the EHP sequence, *Contemp. Math.* **37**(1985), 115–119.

[M6] M.E. Mahowald, The metastable homotopy of S^n, *Memior AMS,* **72**(1967).

[M7] M.E. Mahowald, Some Whitehead products in S^n, *Topology* **4**(1965), 17–26.

[MR] M.E. Mahowald and D.C. Ravenel, Toward a global understanding of the homotopy groups of spheres, Proceedings of the Lefschetz Conference, Mexico, 1984.

[MRay] M.E. Mahowald and N. Ray, A note on the Thom isomorphism, *Proc. Amer. Math. Soc.* **82**(1981), 307–308.

[Mi1] R.J. Milgram, Group representations and the Adams spectral sequence, *Pacific J. Math.* **41**(1972), 157–182.

[Mill] H.R. Miller, On Jones's Kahn–Priddy theorem, to appear.

[MRW] H.R. Miller, D.C. Ravenel and W.S. Wilson, Periodic phenomona in the Adams–Novikov spectral sequence, *Ann. of Math.* **106**(1977), 469–516.

[R1] D.C. Ravenel, *Complex complex cobordism and stable homotopy groups of spheres*, Academic Press, 1986.

[R2] D.C. Ravenel, The nonexistence of odd primary Arf invariant elements in stable homotopy, *Math. Proc. Camb. Phil. Soc.* **83**(1978), 429–443.

[R3] D.C. Ravenel, Localization with respect to certain periodic homology theories, *Amer. J. Math.* **106**(1984), 351–414.

[Sel] P. Selick, *Inv. Math.* **64**(1981), 15–24.

[Sh] P. Shick, Periodic phenomena in the classical Adams spectral sequence, Thesis. Northwestern University, 1984.

[S] V.P. Snaith, Stable decompositions of $\Omega^n \Sigma^n X$, *J. London Math. Soc.* **7**(1974), 577–583.

[T1] H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies No. 49, Princeton University Press, Princeton, 1962.

[T2] T. Kambe, H. Matsunaga and H. Toda, A note on stunted lens spaces, *J. Math. Kyoto Univ.* **5**(1966), 143–149.

[Th] R.D. Thompson, The v_1–periodic homotopy groups of an unstable sphere at odd primes, *Trans. Amer. Math. Soc.* **319**(1990), 535–559.