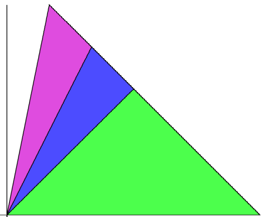
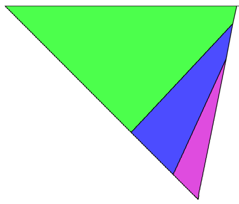
Lecture 1

A solution to the
Arf-Kervaire invariant
problem

AMS Special Session
on Homotopy Theory

October 25, 2009





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1 Background and history

1.1 Our main result

Our main result

Our main theorem can be stated in three different but equivalent ways:

- Manifold formulation: It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds M is always zero.
- Stable homotopy theoretic formulation: It says that certain long sought hypothetical maps between high dimensional spheres do not exist.
- Unstable homotopy theoretic formulation: It says something about the EHP sequence (to be defined below), which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved. 1.3

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Our main result (continued)

Here is the stable homotopy theoretic formulation.

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2+n}(S^n)$ for large n do not exist for $j \geq 7$.*

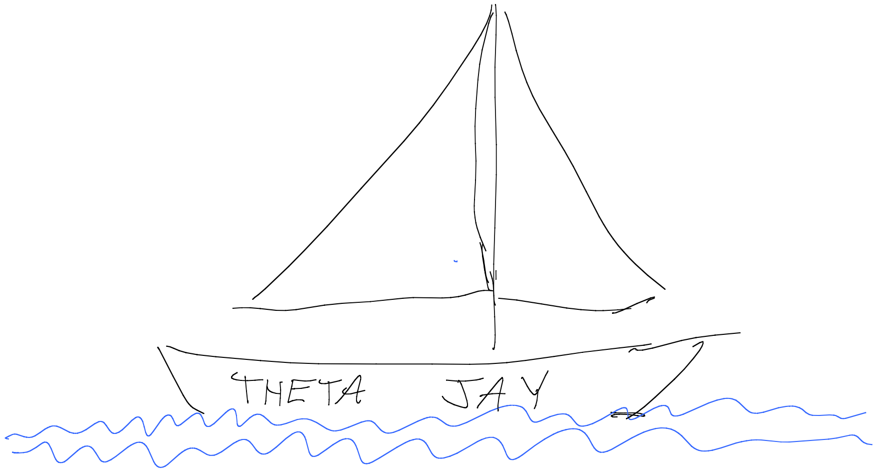
The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2. 1.5

Our main result (continued)

Some homotopy theorists, most notably Mark Mahowald, speculated about what would happen if θ_j existed for all j. They derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the *Doomsday Hypothesis*.

After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then. 1.6

Mark Mahowald's sailboat



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1.2 The Arf-Kervaire formulation

The Arf invariant of a quadratic form in characteristic 2

Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form $\{a_i, b_i \colon 1 \leq i \leq n\}$ with

$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

A *quadratic refinement of* λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying

$$q(x+y) = q(x) + q(y) + \lambda(x,y)$$

Its *Arf invariant* is

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2.$$

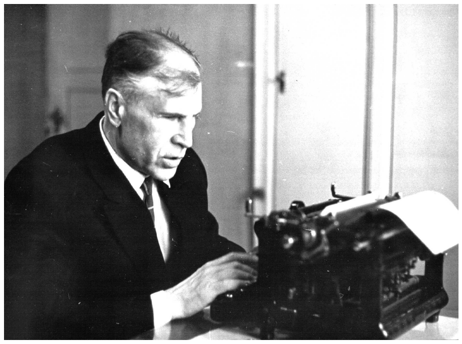
In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q. 1.8

On the money: Arf's definition republished in 2009

1.9

The Kervaire invariant of a framed $(4k+2)$-manifold

Let M be a $2k$-connected smooth closed framed manifold of dimension $4k+2$. The word *framed* here means that M has an embedding in some Euclidean space $\mathbf{R}^{n+4k+2}$ having trivial normal bundle with a given trivialization. This framing leads to a map $p(M) : S^{n+4k+2} \longrightarrow S^n$ and hence an element in $\pi_{n+4k+2}(S^n)$. This construction is due to Pontryagin.



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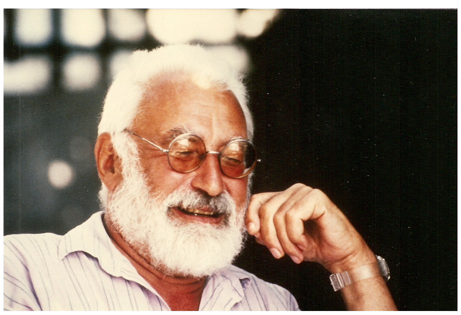
The Kervaire invariant of a framed $(4k+2)$-manifold (continued)

Let $H = H_{2k+1}(M;\mathbf{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an immersion $i_x : S^{2k+1} \looparrowright M$ with a stably trivialized normal bundle. H has an antisymmetric bilinear form λ defined in terms of intersection numbers. Kervaire defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle.



The *Kervaire invariant* $\Phi(M)$ is defined to be the Arf invariant of q.

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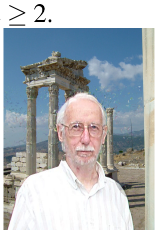
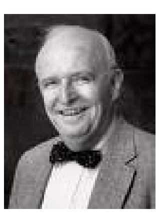
The Kervaire invariant of a framed $(4k+2)$-manifold (continued)

What can we say about $\Phi(M)$?

- Kervaire (1960) showed it must vanish when $k = 2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.
- For $k = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant. Pontryagin used it in 1950 (after some false starts in the 30s) to show $\pi_{n+2}(S^n) = \mathbf{Z}/2$ for all $n \geq 2$.
-   Brown-Peterson (1966) showed that it vanishes for all positive even k.

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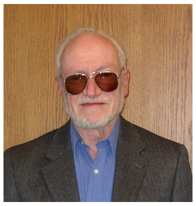
The Kervaire invariant of a framed $(4k+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

-  Browder (1969) showed that it can be nontrivial only if $k = 2^{j-1} - 1$ for some positive integer j. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence. The corresponding element in $\pi_{n+2^{j+1}-2}(S^n)$ for large n is θ_j, the subject of our theorem. *This is the stable homotopy theoretic formulation of the problem.*
- θ_j is known to exist for $1 \le j \le 5$, i.e., in dimensions 2, 6, 14, 30 and 62.
- Our theorem says θ_j does *not* exist for $j \ge 7$. The case $j = 6$ is still open.

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1.3 The unstable formulation

The EHP sequence

Assume all spaces in sight are localized and the prime 2. For each $n > 0$ there is a fiber sequence due to James,

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.$$

This leads to a long exact sequence of homotopy groups

$$\cdots \to \pi_m(S^n) \xrightarrow{E} \pi_{m+1}(S^{n+1}) \xrightarrow{H} \pi_{m+1}(S^{2n+1}) \xrightarrow{P} \pi_{m-1}(S^n) \to \cdots$$

1.14

The EHP sequence (continued)

$$\cdots \to \pi_m(S^n) \xrightarrow{E} \pi_{m+1}(S^{n+1}) \xrightarrow{H} \pi_{m+1}(S^{2n+1}) \xrightarrow{P} \pi_{m-1}(S^n) \to \cdots$$

Here

E stands for Einhängung, the German word for suspension.

H stands for Hopf invariant.

P stands for Whitehead product.

1.15

The EHP sequence (continued)

For $m = 2n$ the sequence is

$$\cdots \to \pi_{2n}(S^n) \xrightarrow{E} \pi_{2n+1}(S^{n+1}) \xrightarrow{H} \underset{\displaystyle \mathbf{Z}}{\underset{\|}{\pi_{2n+1}(S^{2n+1})}} \xrightarrow{P} \pi_{2n-1}(S^n) \to \cdots$$

and we can ask about the image under P of the generator of $\pi_{2n+1}(S^{2n+1})$. We denote it by $w_n \in \pi_{2n-1}(S^n)$, the *Whitehead square*. The following facts are known about it.

- When n is even, w_n it has infinite order and Hopf invariant two.
- w_n is trivial for $n = 1$, 3 and 7. In these cases $w_{n+1} \in \pi_{2n+1}(S^{n+1})$ is divisible by 2, the quotient having Hopf invariant one.
- For other odd values of n, $H(w_{n+1}) = 2$ and w_{n+1} is not divisible by 2, so w_n has order 2.
- For such n, w_n is divisible by 2 iff $n = 2^{j+1} - 1$ with $j > 2$ and θ_j exists, in which case $w_n = 2\theta_j$.

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The Hopf-Whitehead J homomorphism

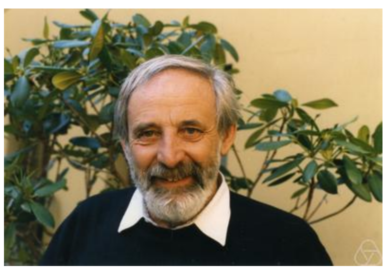
 

Let $SO(n)$ denote the special orthogonal group acting on $\mathbf{R}^n$. Using the one point compactification, each element $g \in SO(n)$ induces a base point preserving map $S^n \to S^n$. Thus we get a map $J : SO(n) \to \Omega^n S^n$ and for each $k > 0$ a homomorphism

$$\pi_k(SO(n)) \xrightarrow{J} \pi_k(\Omega^n S^n) = \pi_{n+k}(S^n).$$

Both source and target known to be independent of n for $n > k+1$. 1.17

The Hopf-Whitehead J homomorphism (continued)



In this case its value for each k was determined by Bott in his periodicity theorem. He showed

$$\pi_k(SO) = \begin{cases} \mathbf{Z} & \text{for } k \equiv 3 \text{ or } 7 \bmod 8 \\ \mathbf{Z}/2 & \text{for } k \equiv 0 \text{ or } 1 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

k	1	2	3	4	5	6	7	8	9	10
$\pi_k(SO)$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0

1.18

The Hopf-Whitehead J homomorphism (continued)

k	1	2	3	4	5	6	7	8	9	10
$\pi_k(SO)$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0

In each case where the group is nontrivial, its generator is known to have nontrivial image (and to generate a direct summand) under J. In the jth case we denote this image by β_j and its dimension by $\phi(j)$, which is roughly $2j$. The first three of these are the Hopf maps $\eta \in \pi_1$, $\nu \in \pi_3$ and $\sigma \in \pi_7$. After that we have $\beta_4 \in \pi_8$, $\beta_5 \in \pi_9$, $\beta_6 \in \pi_{11}$ and so on. Here π_k is short for $\pi_{k+n}(S^n)$ for $n > k+1$, which is known to be independent of n. 1.19

The Hopf-Whitehead J homomorphism (continued)

Each Whitehead square $w_{2n+1} \in \pi_{4n+1}(S^{2n+1})$ (except the cases $n = 0$, 1 and 3) desuspends to a lower sphere until we get an element with a nontrivial Hopf invariant, which is always some β_j. More precisely we have

$$H(w_{(2s+1)2^j-1}) = \beta_j$$

for each $j > 0$ and $s \geq 0$. This result is essentially Adams' 1961 solution to the vector field problem.

1.20

Back to the EHP sequence

Recall the EHP sequence

$$\cdots \to \pi_m(S^n) \xrightarrow{E} \pi_{m+1}(S^{n+1}) \xrightarrow{H} \pi_{m+1}(S^{2n+1}) \xrightarrow{P} \pi_{m-1}(S^n) \to \cdots$$

Given some $\beta_j \in \pi_{2n+1+\phi(j)}(S^{2n+1})$ for $\phi(j) < 2n$, one can ask about the Hopf invariant of its image under P, which vanishes when β_j is in the image of H. In most cases the answer is known and is due to Mahowald. The remaining cases have to do with θ_j. The answer that he had hoped for is the following.

World Without End Hypothesis (Mahowald 1967).

- *The Arf-Kervaire element $\theta_j \in \pi_{2^{j+1}-2}$ exists for all $j > 0$.*
- *It desuspends to $S^{2^{j+1}-1-\phi(j)}$ and its Hopf invariant is β_j.*
- *Let $j, s > 0$ and suppose that $m = 2^{j+2}(s+1) - 4 - \phi(j)$ and $n = 2^{j+1}(s+1) - 2 - \phi(j)$. Then $P(\beta_j)$ has Hopf invariant θ_j.*

1.21

1.4 Questions raised by our theorem

Questions raised by our theorem

EHP sequence formulation. The World Without End Hypothesis was the nicest possible statement of its kind given all that was known prior to our theorem. Now we know it cannot be true since θ_j does not exist for $j \geq 7$. This means the behavior of the indicated elements $P(\beta_j)$ for $j \geq 7$ is a mystery.

Adams spectral sequence formulation. We now know that the h_j^2 for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials. We have no idea what their targets are.

Our method of proof offers a new tool for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future. 1.22

2 Our strategy

2.1 Ingredients of the proof

Ingredients of the proof

Our proof has several ingredients.

- It uses methods of stable homotopy theory, which means it uses spectra instead of topological spaces. The definition of these would take us too far afield, so instead we offer a slogan:

Spectra are to spaces as integers are to natural numbers.

In particular, recall that a space X has a homotopy group $\pi_k(X)$ for each positive integer k. A spectrum X has an abelian homotopy group $\pi_k(X)$ defined for every integer k.

For the sphere spectrum S^0, $\pi_k(S^0)$ is the usual homotopy group $\pi_{n+k}(S^n)$ for $n > k+1$. The hypothetical θ_j is an element of this group for $k = 2^{j+1} - 2$.

1.23

Ingredients of the proof (continued)

More ingredients of our proof:

- It uses complex cobordism theory. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Novikov and Quillen in the 60s. A pivotal tool in the subject is the theory of formal group laws.
- It also makes use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbf{Z}$, but by $RO(G)$, the real representation ring of G. Our calculations make use of this richer structure.

1.24

2.2 The spectrum Ω

The spectrum Ω

We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.

(iii) Gap Theorem. $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.

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The spectrum Ω (continued)

Here again are the properties of Ω

(i) Detection Theorem. If θ_j exists, it has nontrivial image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.

(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$.

(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$. 1.26

2.3 How we construct Ω

How we construct Ω

Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The fixed point set of this action is the set of real points, known to topologists as MO, the unoriented cobordism spectrum. In this notation, U and O stand for the unitary and orthogonal groups. 1.27

How we construct Ω (continued)

To get a C_8-spectrum, we use the following general construction for getting from a space or spectrum X acted on by a group H to one acted on by a larger group G containing H as a subgroup. Let

$$Y = \mathrm{Map}_H(G, X),$$

the space (or spectrum) of H-equivariant maps from G to X. Here the action of H on G is by right multiplication, and the resulting object has an action of G by left multiplication. As a set, $Y = X^{|G/H|}$, the $|G/H|$-fold Cartesian power of X. A general element of G permutes these factors, each of which is left invariant by the subgroup H.

In particular we get a C_8-spectrum

$$MU^{(4)} = \mathrm{Map}_{C_2}(C_8, MU).$$

This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is. 1.28