The Periodicity Theorem in the solution to the Arf-Kervaire invariant problem

Princeton University Workshop on the Kervaire Invariant

February 11, 2010

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University of Virginia
Mike Hopkins
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Doug Ravenel
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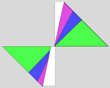
Review of our strategy

Our goal is to prove

Main Theorem

The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$.



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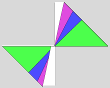

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(iii) It is 256-periodic, meaning $\Sigma^{256}\Omega \cong \Omega$. This is the Periodicity Theorem.

Our strategy (continued)

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(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

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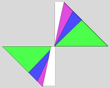

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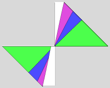

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The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

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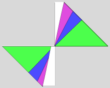
The spectrum Ω

As explained previously, there is an action of the cyclic group C_8 on the 4-fold smash product $MU^{(4)}$.



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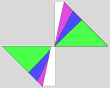

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The spectrum Ω (continued)

The homotopy of $(MU^{(4)})^{hC_8}$ can be computed using the homotopy fixed point spectral sequence, for which

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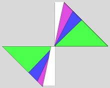

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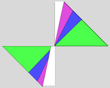

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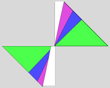

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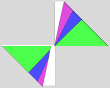
In order to identify D we need to study the slice spectral sequence in more detail.



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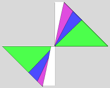

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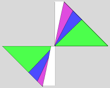
${}^GP_n^nMU^{(4)}$ is the nth slice and the decreasing sequence of subgroups of $\pi_*(MU^{(4)})$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_\star(MU^{(4)})$ and the homotopy groups of fixed point sets $\pi_*((MU^{(4)})^H)$ for each subgroup H.

The slice spectral sequence (continued)

This means the slice filtration leads to a slice spectral sequence converging to $\pi_*(MU^{(4)})$ and its variants.



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One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^G P_t^t MU^{(4)}) \implies \pi_{t-s}^G(MU^{(4)}).$$

Recall that $\pi_*^G(MU^{(4)})$ is by definition $\pi_*((MU^{(4)})^G)$, the homotopy of the fixed point set.

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In the slice tower for $MU^{(4)}$, every odd slice is contractible and $P_{2n}^{2n} = \hat{W}_n \wedge H\mathbf{Z}$, where $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum and $\hat{W}_n$ is a certain wedge of the following three types of finite G-spectra:

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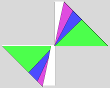

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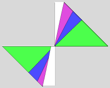

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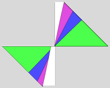
The same holds after we invert D, in which case negative values of n can occur.



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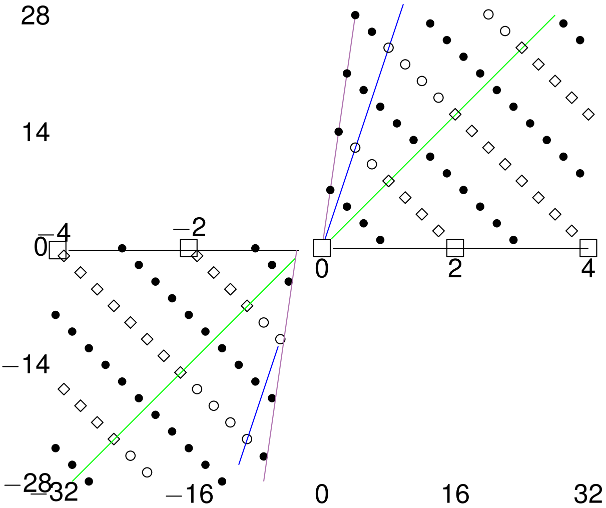

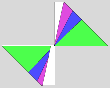

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- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7,

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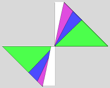

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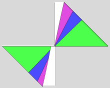

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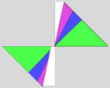
- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.



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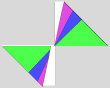

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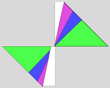

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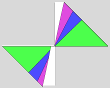

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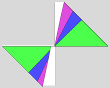

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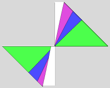

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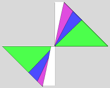

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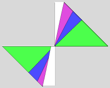
  means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture.



Geometric fixed points

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Unstably a G-space X has a fixed point set,

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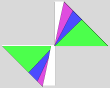

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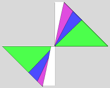

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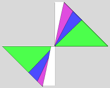
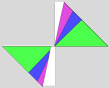


Geometric fixed points (continued)

Both of these definitions have stable analogs, but the fixed point functor is awkward for two reasons:



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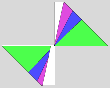

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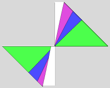

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Here EC_2 is a G-space via the projection $G \to C_2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a G-space.

Geometric fixed points (continued)

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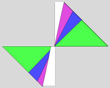
Under this action EC_2^G is empty while for any proper subgroup H of G, $EC_2^H = EC_2$, which is contractible.



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Definition

For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

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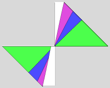

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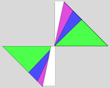

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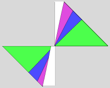

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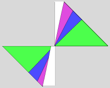

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Geometric Fixed Point Theorem

Let σ denote the sign representation. Then for any G-spectrum X, $\pi_\star(\tilde{E}C_2 \wedge X) = a_\sigma^{-1}\pi_\star(X)$, where $a_\sigma : S^0 \to S^\sigma$ is the inclusion of the fixed point set.

Geometric fixed points (continued)

Recall that $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$.

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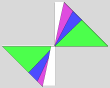
It is not hard to show that

$$\pi_*(MU^{(4)}) = \mathbf{Z}[r_i, \gamma(r_i), \gamma^2(r_i), \gamma^3(r_i) : i > 0]$$

where $|r_i| = 2i$, γ is a generator of G and $\gamma^4(r_i) = (-1)^i r_i$.



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$$Nr_i = r_i\gamma(r_i)\gamma^2(r_i)\gamma^3(r_i).$$

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Lemma

The generators r_i and y_i can be chosen so that

$$\Phi^G Nr_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

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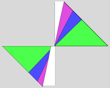
Some slice differentials

We know that the slice spectral sequence for $MU^{(4)}$ has a vanishing line of slope 7.



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Let $f_i \in \pi_i(MU^{(4)})$ be the composite

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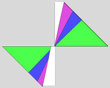

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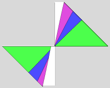

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- The subring of elements on the vanishing line is the polynomial algebra on the f_i.

Some slice differentials (continued)

- Under the map

$$\pi_*(MU^{(4)}) \to \pi_*(\Phi^G MU^{(4)}) = \pi_*(MO)$$

we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$

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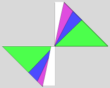
Recall that for an oriented representation V there is a map $u_V : S^{|V|} \to \Sigma^V H\mathbf{Z}$, which lies in $\pi_{V-|V|}(H\mathbf{Z})$.



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Recall that for an oriented representation V there is a map $u_V : S^{|V|} \to \Sigma^V H\mathbf{Z}$, which lies in $\pi_{V-|V|}(H\mathbf{Z})$. It satisfies $u_{2V} = u_V^2$, so $u_{2^k\sigma} = u_{2\sigma}^{2^{k-1}}$.

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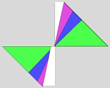

Some $RO(G)$-graded calculations

For a cyclic 2-group G let

$$\begin{aligned}\overline{\Delta}_k^{(g)} = N_2^g r_{2^k-1} &= r_{2^k-1}\gamma(r_{2^k-1})\ldots\gamma^{g/2-1}(r_{2^k-1}) \\ &\in \pi_{(2^k-1)\rho_g}(MU^{(g/2)})\end{aligned}$$

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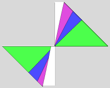

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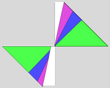

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The differential d_r on $u_{2\sigma}^{2^k}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line.

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The differential d_r on $u_{2\sigma}^{2^k}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{\Delta}_k^{(g)}$, then $u_{2\sigma}^{2^k}$ will be a permanent cycle.

Some $RO(G)$-graded calculations (continued)

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We have

$$f_{2^{k+1}-1}\overline{\Delta}_k^{(g)} = (a_{\rho_g}^{2^{k+1}-1} N r_{2^{k+1}-1})(N r_{2^k-1})$$

Some $RO(G)$-graded calculations (continued)

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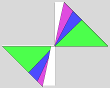

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Some $RO(G)$-graded calculations (continued)

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Corollary

In the $RO(G)$-graded slice spectral sequence for $\left(\overline{\Delta}_k^{(g)}\right)^{-1} MU^{(g/2)}$, the class $u_{2^{k+1}\sigma} = u_{2\sigma}^{2^k}$ is a permanent cycle.

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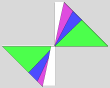
An even trickier $RO(G)$-graded calculation

The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle.



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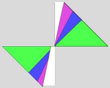
We will get this by using the norm property of u.



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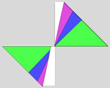

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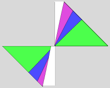

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From this we can deduce that $u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2})$,

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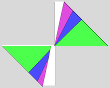
An even trickier $RO(G)$-graded calculation (continued)

We have $u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2})$.



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By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\Delta}_{k_m}^{(2^m)}$ for $1 \leq m \leq 3$.

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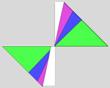

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An even trickier $RO(G)$-graded calculation (continued)

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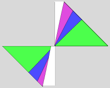

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- Inverting $\overline{\Delta}_1^{(8)}$ makes $u_{4\sigma_8}$ a permanent cycle.

An even trickier $RO(G)$-graded calculation (continued)

We have $u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2})$.

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- Inverting $\overline{\Delta}_2^{(4)}$ makes $u_{8\sigma_4}$ a permanent cycle.
- Inverting $\overline{\Delta}_1^{(8)}$ makes $u_{4\sigma_8}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8} = u_{2\rho_8}^{16}$ a permanent cycle.

An even trickier $RO(G)$-graded calculation (continued)

Let

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)}) \in \pi_{19\rho_8}(MU^{(4)}).$$

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An even trickier $RO(G)$-graded calculation (continued)

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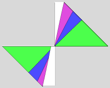

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Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(\Omega) = \pi_*^{C_8}(\tilde{\Omega})$ has the usual properties:

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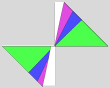
- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.

An even trickier $RO(G)$-graded calculation (continued)

Let
$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)}) \in \pi_{19\rho_8}(MU^{(4)}).$$
The we define $\tilde{\Omega} = D^{-1} MU^{(4)}$ and $\Omega = \tilde{\Omega}^{C_8}$.

Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(\Omega) = \pi_*^{C_8}(\tilde{\Omega})$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0.



The proof of the Periodicity Theorem

Preperiodicity Theorem

Let $\Delta_1^{(8)} = u_{2\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^2 \in E_2^{16,0}(D^{-1}MU^{(4)}) = E_2^{16,0}(\tilde{\Omega})$. *Then* $\left(\Delta_1^{(8)}\right)^{16}$ *is a permanent cycle.*

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To prove this, note that $\left(\Delta_1^{(8)}\right)^{16} = u_{32\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^{32}$.

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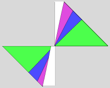
- $\Delta_1^{(8)}$ is invertible because it is a factor of D.

The proof of the Periodicity Theorem (continued)

The above imply that the underlying map $i_0\Pi$ of ordinary spectra is a homotopy equivalence.



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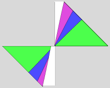

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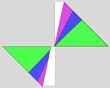

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Periodicity Theorem

Let $\Omega = (D^{-1}MU^{(4)})^{C_8}$. Then $\Sigma^{256}\Omega$ is equivalent to Ω.

Recap of the proof

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- $\tilde{\Omega}$ is obtained from the C_8-spectrum $MU^{(4)}$ by inverting a certain element

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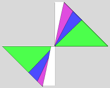

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Recap of the proof (continued)

- The resulting equivariant map

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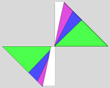
is an equivalence of the underlying spectra.



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- $\pi_*(\tilde{\Omega}^{hC_8})$ is accessible via the Adams-Novikov spectral sequence, and we know that it detects each θ_j, in addition to being 256-periodic.
- Our Homotopy Fixed Point Theorem (not covered in this talk) equates $\tilde{\Omega}^{hC_8}$ with $\Omega = \tilde{\Omega}^{C_8}$, which is known to have the gap property.