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An overview of Akhmet'ev's program

Let $n = 2^{j+1} - 2$ for some positive integer j. Browder's theorem tells us that the Arf-Kervaire invariant in dimension n is related to the element h_j^2 in the Adams spectral sequence. There is a famed n-manifold with Arf-Kervaire invariant one if and only if h_j^2 is a permanent cycle. Using the Kahn-Priddy theorem we can pull this back to a similar statement about the Adams spectral sequence for $\pi_*(\mathbf{R}P^\infty)$. There is a map from this Adams spectral sequence to the one for the sphere spectrum which raises filtration by one. We can say that an element in $\pi_n(\mathbf{R}P^\infty)$ has nontrivial Arf-Kervaire invariant if it is detected by h_j^2.

Now suppose $f : M^{n-1} \looparrowright \mathbf{R}^n$ is a codimension one immersion of a (not necessarily framed) manifold M. Its normal bundle is a line bundle λ classified by a map $M^{n-1} \to \mathbf{R}P^\infty$. If we compose f with the inclusion $\mathbf{R}^n \hookrightarrow \mathbf{R}^{n+t}$ for large enough t, it becomes regularly homotopic to an embedding. Then we can use the Pontrjagin-Thom construction to get a map $S^{n+t} \to \Sigma^t MO(1) = \mathbf{R}P^\infty$, that is a stable map $S^n \to \mathbf{R}P^\infty$. Thus $\pi_n(\mathbf{R}P^\infty)$ can be identified with the cobordism group of codimension 1 immersions in $\mathbf{R}^n$.

Associated with the immersion f is the set of double points

$$N^{n-2} = \{\{x, y\} \subset M \colon f(x) = f(y) \text{ and } x \neq y\},$$

the set of unordered pairs of distinct points in M having the same image in $\mathbf{R}^n$. It is doubly covered by a similar set $\tilde{N}^{n-2}$ or ordered pairs of distinct points in M. A small perturbation of f will make N a codimension one submanifold of the orbifold SP^2M. The immersion f induces a codimension 2 immersions $g : N^{n-2} \looparrowright \mathbf{R}^n$ and $\tilde{g} : \tilde{N}^{n-2} \looparrowright \mathbf{R}^n$. The normal 2-plane bundles η and $\tilde{\eta}$ of these immersions have structure groups D_4 (the dihedral group of order 8) and $(\mathbf{Z}/2)^2$ respectively.

Eccles has identified the Arf-Kervaire invariant of f with the characteristic number

$$\langle w_2(\eta)^{(n-2)/2}, [N^{n-2}]\rangle.$$

Hence if this could be shown to vanish for all f, then we would know that θ_j does not exist. He has a similar statement about the Hopf invariant, for which $n = 2^j - 1$ and the relevant characteristic number is

$$\langle w_1(\eta)^{n-2}, [N^{n-2}]\rangle.$$

This construction can be generalized in two different ways:

(i) For an integer $k > 1$, let $M^{n-k} \subset M^{n-1}$ be a codimension $(k-1)$-submanifold dual to $w_1(\lambda)^{k-1}$. The restriction of f to M^{n-k} is a codimension k immersion with normal bundle isomorphic to $k\lambda$. This suggests the following definition: A codimension k *skew framed immersion* is a triple (f, Ξ, κ) where $f : M^{n-k} \looparrowright \mathbf{R}^n$ is a codimension k immersion, κ is a line bundle over M and Ξ is an isomorphism of the normal bundle ν_f with $k\kappa$. Such an immersion represents an element in $\pi_n(\mathbf{R}P^\infty)$ even if it did not come from a codimension one immersion as described above.

Passing to double points gives us a triple (g, Ψ, η) where $g : N^{n-2k} \looparrowright \mathbf{R}^n$ is a codimension $2k$ immersion, η is a 2-plane bundle with structure group D_4, and Ψ is an isomorphism between the normal bundle ν_g and $k\eta$. It is still the case that the Arf-Kervaire invariant of f is the characteristic number

$$\langle w_2(\eta)^{(n-2k)/2}, [N^{n-2k}]\rangle.$$

There is a similar formuala in the Hopf invariant case.

(ii) We can iterate the double point construction to obtain quadruple points, octuple points, and so on. In the codimension one case, we get an immension $h : N^{n-2^s} \looparrowright \mathbf{R}^n$. The structure group of its normal bundle is the 2-Sylow subgroup of the symmetric group $\Sigma_{2^{s+1}}$, which Akhmet'ev denotes by $\mathbf{Z}/2^{[s+1]}$. In the codimension k case, we get a triple (h, Λ, ζ), where $h : L^{n-2^s j} \looparrowright \mathbf{R}^n$ is a codimension $2^s k$ immersion, ζ is a 2^s-plane bundle with structure group $\mathbf{Z}/2^{[s+1]}$, and Λ is an isomorphism between the normal bundle ν_h and $k\zeta$. It is still the case that the Arf-Kervaire invariant is the characteristic number

$$\langle w_2(\eta)^{(n-2^s k)/2}, [L^{n-2^s k}]\rangle.$$

Now things get more difficult. Akhmet'ev needs to show that under certain conditions the structure group $\mathbf{Z}/2^{[s+1]}$ can be replaced by a smaller subgroup, which is then shown to imply that the Arf-Kervaire or Hopf invariant vanishes. In the latter case one needs to reduce from $\mathbf{Z}/2^{[2]} = D_4$ to $\mathbf{Z}/4$. This means that at the next stage we have a reduction from $\mathbf{Z}/2^{[3]} = \mathbf{Z}/2^{[2]} \wr \mathbf{Z}/2$ to $\mathbf{Z}/4 \wr \mathbf{Z}/2$, and we need to reduce further to the quaternion group Q_8.

In the Arf-Kervaire case, one needs to reduce from $\mathbf{Z}/2^{[6]}$ (which has order 2^{63}) to $Q_8 \times Q_8$ (with order 2^6). According to his Princeton talk this is accomplished by successive reductions of each $\mathbf{Z}/2^{[s+1]}$ for $s < 5$ to a certain subgroup. *Thus far, I have not been able to follow any of these proofs.*

In each case the condition which enables one to reduce the structure group has the following form. The original skew framed immersion (up to cobordism) corrsponds to an element in $\pi_n(\mathbf{R}P^\infty)$. Since n is even, this group is the same as $\pi_n(\mathbf{R}P^n)$. The hypothesis needed is that the map $S^n \to \mathbf{R}P^n$ factors through $\mathbf{R}P^{n-q}$ for a suitable integer $q > 0$. The following result guarantees that this condition can be met for large enough j.

Desuspension Theorem. *For each positive integer q there is an integer $j(q)$ such that for any $j \geq j(q)$, each element in $\pi_n(\mathbf{R}P^n)$ (where $n = 2^{j+1} - 2$) is in the image of $\pi_n(\mathbf{R}P^{n-q})$.*

The proof of this theorem and possible values of $j(q)$ are discussed elsewhere on this website. I know of no way in general to estimate $j(q)$, but for $q \leq 15$ the growth of $j(q)$ appears to exponential.