THE TRIPLE LOOP SPACE APPROACH TO THE TELESCOPE CONJECTURE

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The purpose of this paper is to describe an *unsuccessful* attempt to prove that the telescope conjecture (see 1.13 below for the precise statement) is false for $n \geq 2$ and each prime p. At the time the it was originally formulated over 20 years ago (see [Rav84]), the telescope conjecture appeared to be the simplest and most plausible statement about the question at hand, namely the relation between two different localization functors. We hope the present paper will demonstrate that this is no longer the case. We will set up a spectral sequence converging to the homotopy of one of the two localizations (the geometrically defined telescope) of a certain spectrum, and it will be apparent that only a bizarre pattern of differentials would lead to the known homotopy of the localization defined in terms of BP-theory, the answer predicted by the telescope conjecture. While we cannot exclude such a pattern, it is certainly not favored by Occam's razor.

No use will be made here made of the parametrized Adams spectral sequence of [Rav92b]; we will say more about that approach in a future paper. Instead we will rely on some constructions related to the EHP sequence which are described in §3, where we define the spectra $y(n)$ and $Y(n)$, and a variant of the Eilenberg-Moore spectral sequence (which we call the Thomified Eilenberg-Moore spectral sequence) described in §2.4.

§1 is an expository introduction to the telescope conjecture. We define telescopes and recall the nilpotence (1.1), periodicity (1.4) and thick subcategory (1.12) theorems of Devinatz, Hopkins and Smith ([DHS88] and [HS98]). We also recall the definitions of Bousfield localization and related concepts and the Bousfield localization theorem (1.8). We then state four equivalent formulations of the telescope conjecture in 1.13.

In §2 we introduce the various spectral sequences that we will use. These include the classical Adams (§2.1) and Adams-Novikov (§2.2) spectral sequences. We also need the localized Adams spectral sequence of Miller [Mil81] (§2.3), for which we prove a convergence theorem 2.13. This is the spectral sequence we will use to compute the homotopy of our telescope $Y(n)$ and see that it may well differ from the answer predicted by the telescope conjecture. In §2.4 we introduce the Thomified Eilenberg-Moore spectral sequence and its localized form. In certain cases (2.26 and 2.27) we identify its E_2-term as Ext over a Massey-Peterson algebra. All of these spectral sequences require the use of

Ext groups over various Hopf algebras, and we review the relevant homological algebra in §2.5. This includes two localizations ((2.34) and (2.35)) of the Cartan-Eilenberg spectral sequence which are new as far as we know.

In §3 we use the EHP sequence to construct the spectrum $y(n)$ and its telescope $Y(n)$. We describe the computation of $\pi_*(L_n y(n))$ using the Adams-Novikov spectral sequence, and then state our main computational conjecture, 3.16, which says that the localized Adams spectral sequence gives a different answer for $\pi_*(Y(n))$ when $n > 1$. This would disprove the telescope conjecture, which predicts that $L_n y(n) = Y(n)$. The conjectured difference between $\pi_*(L_n y(n))$ and $\pi_*(Y(n))$ can be described very simply: $\pi_*(L_n y(n))$ is finitely generated as a module over the ring $K(n)_*[v_{n+1}, v_{n+2}, \ldots, v_{2n}]$, whereas, if our main conjecture is correct, then $\pi_*(Y(n))$ will have no finite presentation over this ring.

Our construction of $y(n)$ gives us a map

$$\Omega^3 S^{1+2p^n} \xrightarrow{f} y(n),$$

with which we originally hoped to prove Conjecture 3.16 and is the reason for the title of this paper. In §4 we recall some properties $\Omega^3 S^{1+2p^n}$, including the Snaith splitting (4.2) and its ordinary homology as a module over the Steenrod algebra (Lemma 4.7). In §4.3 we recall Tamaki's unpublished computation of its Morava K-theory using his formulation [Tam94] of the Eilenberg-Moore spectral sequence, and in §4.4 we show that similar methods can be used to compute its $Y(n)_*$-theory. These are not needed for our main results and are included due to their independent interest.

In §5 we describe our program to prove Conjecture 3.16 and thereby disprove the telescope conjecture for $n > 1$. Our method is to construct a map (derived from the map f above) to the localized Adams spectral sequence for $Y(n)_*$ from a localized Thomified Eilenberg-Moore spectral sequence converging to $Y(n)_*(\Omega^3 S^{1+2p^n})$. This map turns out to be onto in each E_r, so differentials in the latter spectral sequence are determined by those in the former, which are described in Conjecture 5.15. The source spectral sequence has far more structure than the target, and we had hoped to use this to prove 5.15. There are three such structures, each of which figures in the program, namely:

(i) $\Omega^3 S^{1+2p^n}$ is an H-space, so the spectral sequence is one of Hopf algebras.
(ii) It has a Snaith splitting which must be respected by differentials.

(iii) The pth Hopf map induces an endomorphism of our spectral sequence, which is identified in Lemma 5.16.

Previously we had thought that this structure could be used to construct certain permanent cycles $\tilde{y}_{i,j}$ mapping to $b_{n+i,j}$ in the localized Adams spectral sequence that would force the latter to collapse from a certain stage. Unfortunately, this is not the case. For more details, see the comments after Conjecture 5.12.

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1. The telescope conjecture and Bousfield localization

1.1. **Telescopes.** The telescope conjecture is a statement about the stable homotopy groups of finite complexes. There is not a single nontrivial example for which such groups are completely known. There are many partial results, especially about the stable homotopy groups of spheres. Unstably the situation is only slightly better. We have complete knowledge of $\pi_*(X)$ for a finite complex X only in the cases where X is known to be an Eilenberg-Mac Lane space, such as when X is a surface of positive genus.

Experience has shown that one can get interesting information about $\pi_*(X)$ in the stable case in the following way. Suppose one has a stable map of the form

$$\Sigma^d X \xrightarrow{f} X$$

for which all iterates are essential; this can only happen if $d \geq 0$. Such a map is said to be *periodic.* We say that f is *nilpotent* if some iterate of it is null. In any case we can define the *telescope* $f^{-1}X$ to be the direct limit of the system

$$X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{f} \Sigma^{-2d} X \xrightarrow{f} \cdots .$$

This will be contractible if f is nilpotent. In the (rare) cases when f is periodic, *the computation of* $\pi_*(f^{-1}X)$ *is far more tractable than that of* $\pi_*(X)$.

The map f induces an endomorphism of $\pi_*(X)$, which we will denote abusively by f, making $\pi_*(X)$ a module over the ring $\mathbf{Z}[f]$. Since homotopy commutes with direct limits, we have

$$\pi_*(f^{-1}X) = \pi_*(X) \otimes_{\mathbf{Z}[f]} \mathbf{Z}[f, f^{-1}].$$

The telescope conjecture is a statement about this graded group.

Before stating it we will describe some motivating examples. We assume that all spaces and spectra in sight are localized at a prime p.

- For any spectrum X let f be the degree p map. It induces multiplication by p in homotopy and homology and induces an isomorphism in rational homology. If $H_*(X;\mathbf{Q})$ is nontrivial, i.e., if the integer homology of X is not all torsion, then all iterates of the degree p map are essential.

  In this case the telescope $p^{-1}X$ is the *rationalization* $X\mathbf{Q}$ of X with

$$\pi_*(X\mathbf{Q}) = \pi_*(X) \otimes \mathbf{Q} = H_*(X;\mathbf{Q}),$$

  the rational homotopy of X. It is a rational vector space.
- Let $V(0)$ be the mod p Moore spectrum. For each prime p Adams [Ada66] constructed a map

$$\Sigma^d V(0) \xrightarrow{\alpha} V(0) \quad \text{where} \quad d = \begin{cases} 8 & \text{if } p = 2 \\ 2p-2 & \text{if } p \text{ is odd.} \end{cases}$$

  This map induces an isomorphism in classical K-theory and all iterates of it are nontrivial. $\pi_*(\alpha^{-1}V(0))$ has been computed explicitly by Mahowald [Mah81] for $p = 2$ and Miller [Mil81] for odd primes. It is finitely presented as a module over $\mathbf{Z}[\alpha, \alpha^{-1}]$. The image of $\pi_*(V(0))$ in $\pi_*(\alpha^{-1}V(0))$ is known, and this gives us a lot of information about the former.

  By analogy with the previous example, one might expect $\pi_*(\alpha^{-1}V(0))$ to be $K_*(V(0))$, but the situation here is not so simple. The answer is however predictable by K-theoretic or BP-theoretic methods; we will say more about this later.
- For odd p let $V(1)$ denote the cofiber of the Adams map α. It is a CW-complex with one cell each in dimensions 0, 1, $2p-1$ and $2p$. Smith [Smi71] and Toda [Tod71] have shown that for $p \geq 5$ there is a periodic map

$$\Sigma^{2p^2-2} V(1) \xrightarrow{\beta} V(1).$$

  In this case the homotopy of the telescope is not known.

The results of Devinatz-Hopkins-Smith ([DHS88] and [HS98]) allow us to study telescopes in a very systematic way. They indicate that BP-theory and Morava K-theory are very useful here. First we have the nilpotence theorem characterizing nilpotent maps.

Theorem 1.1 (Nilpotence theorem)**.** *For a finite p-local spectrum X, a map*

$$\Sigma^d X \xrightarrow{f} X$$

is nilpotent if and only if the induced map on $BP_*(X)$ is nilpotent. Equivalently, it is nilpotent if and only if the induced map on $K(n)_*(X)$ is nilpotent for each n.

For the study of periodic maps two definitions are useful.

Definition 1.2. *A p-local finite complex X has* type n *if n is the smallest integer for which $K(n)_*(X)$ is nontrivial.*

Definition 1.3. *A map*

$$\Sigma^d X \xrightarrow{f} X$$

is a v_n-map *if $K(n)_*(f)$ is an isomorphism and $K(m)_*(f) = 0$ for $m \neq n$. (The spectrum X here need not be finite.)*

A finite complex of type n does not admit a v_m-map for $m > n$; this follows from the algebraic properties of the target category of the BP-homology functor. For $m < n$, the trivial map is a v_m-map. The cofiber of a v_n-map on a type n complex is necessarily a complex of type $n+1$. In the three examples above we have a such a map for $n = 0$, 1 and 2 respectively.

Now we can state the periodicity theorem of [HS98].

Theorem 1.4 (Periodicity theorem)**.** *Every type n finite complex admits a v_n-map. Given two such maps f and g there are positive integers i and j such that $f^i = g^j$.*

Corollary 1.5. *For a type n p-local finite complex X, any v_n-map $f : \Sigma^d X \to X$ yields the same telescope $f^{-1}X$, which we will denote by $v_n^{-1}X$ or $\widehat{X}$.*

1.2. Bousfield localization and Bousfield classes.

Definition 1.6. *Given a homology theory h_*, a spectrum X is h_*-local if for each spectrum W with $h_*(W) = 0$, $[W, X] = 0$. An h_*-localization $X \to L_h X$ is an h_*-equivalence from X to an h_*-local spectrum. We denote the fiber of this map by $C_h X$. If h_* is represented by a spectrum E we will write L_E and C_E for L_h and C_h. The case $E = v_n^{-1}BP$ is of special interest, and we denote the corresponding functors by L_n and C_n.*

The following properties of localization are formal consequences of these definitions.

Proposition 1.7. *If $L_h X$ exists it is unique and the functor L_h is idempotent. The map $X \to L_h X$ is terminal among all h_*-equivalences from X and initial among all maps from X to h_*-local spectra. $C_h X$ is h_*-acyclic and the map $C_h X \to X$ is terminal among all maps from*

h_*-acyclics to X. The homotopy inverse limit of h_*-local spectra is h_*-local, although the functor L_h (if it exists) need not commute with homotopy inverse or direct limits. The homotopy direct limit of local spectra need not be local.

The definitive theorem in this subject is due to Bousfield [Bou79].

Theorem 1.8 (Bousfield localization theorem)**.** *The localization L_hX exists for all spectra X and all homology theories h_*.*

Roughly speaking, one constructs C_hX by taking the direct limit of all h_*-acyclic spectra mapping to X. (This is not precisely correct because of set theoretic problems; there are too many such maps to form a direct limit. Bousfield found a way around this difficulty.) A variant on this procedure is to consider the homotopy direct limit of all *finite* h_*-acyclic spectra mapping to X, which we denote by C_h^fX. (Here f stands for finite, and there are no set theoretic problems.) We denote the cofiber of $C_h^fX \to X$ by L_h^fX.

Definition 1.9. *A localization functor L_h is finite if $L_h = L_h^f$, i.e., if C_hX is always a homotopy direct limit of finite h_*-acyclic spectra mapping to X.*

Proposition 1.10. *If the functor L_h is finite then*

(i) *it commutes with homotopy direct limits,*
(ii) *the homotopy direct limit of h_*-local spectra is local,*
(iii) *$L_hX = X \wedge L_hS^0$ for all X, and*
(iv) *L_h is the same as Bousfield localization with respect to the homology theory represented by L_hS^0.*

It can be shown [Rav84, Prop. 1.27] that the four properties listed in 1.10 are equivalent. We say that a localization functor is *smashing* if it has them. Thus 1.10 says that every finite localization functor is smashing. Bousfield conjectured [Bou79, 3.4] the converse, that every smashing localization functor is finite. The functor L_n is known to be smashing [Rav92a, Theorem 7.5.6], but if the telescope conjecture fails, it is not finite for $n \geq 2$.

Definition 1.11. *Two spectra E and F are Bousfield equivalent if they have the same acyclics (i.e. if $E_*(X) = 0$ iff $F_*(X) = 0$,) or equivalently if $L_E = L_F$. The corresponding equivalence class is denoted by $\langle E \rangle$, the Bousfield class of E. We say that $\langle E \rangle \geq \langle F \rangle$ if $E_*(X) = 0$ implies $F_*(X) = 0$.*

Dror Farjoun [Far96] uses the notation $X \ll Y$ (Y can be built from X by cofibrations) in an unstable context to mean $\langle X \rangle > \langle Y \rangle$.

The following consequence of 1.1 is very useful, e.g. it was used to prove 1.4. A subcategory of the stable homotopy category of finite complexes is *thick* if it is closed under cofibrations and retracts. One example is the subcategory of h_*-local finite spectra for a given h. The following result of [DHS88] classifies all thick subcategories.

Theorem 1.12 (Thick subcategory theorem)**.** *Any nontrivial thick subcategory of the stable homotopy category of p-local finite complexes is the category $\mathbf{C}_n$ of p-local finite $K(n-1)_*$-acyclic spectra for some $n \geq 0$.*

Note that $\mathbf{C}_0$ is the entire category of p-local finite spectra,

$$\mathbf{C}_0 \supset \mathbf{C}_1 \supset \mathbf{C}_2 \supset \dots,$$

and the intersection of all these is the trivial subcategory consisting of a point.

This dry sounding theorem is a useful tool. Suppose one wants to prove that all p-local finite spectra of type $\geq n$ satisfy a certain property, say that they are all *demented*. (This example is due to John Harper.) If one can show that the subcategory of demented spectra is thick, then all that remains is to show that a single one of type n is demented. If one is demented they all are demented. Conversely, if we can find a single type n spectrum that is not demented, then none of them are.

1.3. **The telescope conjecture.** Now we will discuss several equivalent formulations of the telescope conjecture.

Telescope conjecture 1.13. *Choose a prime p and an integer $n \geq 0$. Let X be a p-local finite complex of type n (1.2) and let $\widehat{X}$ be the associated telescope (1.5). Then*

(i) $\widehat{X} = L_n X$.
(ii) $\langle \widehat{X} \rangle = \langle K(n) \rangle$.
(iii) *The Adams-Novikov spectral sequence for $\widehat{X}$ converges to $\pi_*(\widehat{X})$.*
(iv) *The functors L_n and L_n^f are the same if $L_{n-1} = L_{n-1}^f$.*

We will sketch the proof that the four statements above are equivalent.

The set of $K(n-1)_*$-acyclic finite p-local spectra satisfying (i) is thick. The same is true for (ii) and for the statement that

$$\langle L_n X \rangle \leq \langle K(n) \rangle. \tag{1.14}$$

Thus if we can find a type n X with this property it will follow that (i) and (ii) are equivalent. One can show that (1.14) holds if the Adams-Novikov E_2-term for X has a horizontal vanishing line; this means that L_nX can be built out of $K(n)$ with a finite number of cofibrations. Such an X can be constructed using the methods described in [Rav92a, §8.3].

For the third statement, the Adams-Novikov spectral sequence for L_nX (which is BP_*-equivalent to $\widehat{X}$) was shown in [Rav87] to converge to its homotopy, so it also converges to that of $\widehat{X}$ iff (i) holds.

For the fourth statement, since the functors L_n and L_n^f are both smashing, they commute with homotopy direct limits. This means that if they agree on finite complexes, they agree on all spectra. For $K(n-1)_*$-acyclic X it is known that $L_n^fX = \widehat{X}$ (see [Rav93b], Miller [Mil92] or Mahowald-Sadofsky [MS95]) so (i) says the two functors agree on such X. For finite p-local X of smaller type, the methods of [Rav93b, §2] show that C_{n-1}^f (the fiber of $X \to L_{n-1}^fX$) is a homotopy direct limit of type n finite complexes, so $L_nX = L_n^fX$.

Any attempt to prove 1.13 is likely to rely on 1.12. It is easy to show that the set of $K(n-1)_*$-acyclic finite spectra satisfying 1.13(i) is thick. Thus one can prove or disprove the telescope conjecture if we can compare $\pi_*(L_nX)$ with $\pi_*(\widehat{X})$ for a single type n spectrum X. The telescope conjecture for $n=1$ follows from the computations of Mahowald [Mah81] and Miller [Mil81] of $\pi_*(\widehat{V(0)})$ which showed that agrees with the previously known value of $\pi_*(L_1V(0))$. Alternately we can disprove the telescope conjecture by finding a spectrum Y (which need not be finite) for which $L_n^fY \neq L_nY$.

The groups $\pi_*(L_nX)$ (or $\pi_*(L_nY)$) and $\pi_*(\widehat{X})$ (or $\pi_*(L_n^fY)$) can be computed with variants of the Adams spectral sequence. These methods will be discussed in the next section.

The spectrum we will use, $y(n)$, is a certain Thom spectrum which will be constructed in §3. We will use the Adams-Novikov spectral sequence to show (Corollary 3.12) that $\pi_*(L_ny(n))$ is finitely generated over a certain ring $R(n)_*$ defined below in (3.13); this is relatively easy. A far more difficult calculation (Conjecture 3.16) using the localized Adams spectral sequence (described in §2.3) comes quite close to showing that $\pi_*(L_n^fy(n))$ is *not* finitely generated over $R(n)_*$ for $n > 1$, which would disprove the telescope conjecture.

1.4. **Some other open questions.** The spectrum $y(n)$ of §3 has a telescope $Y(n)$ associated with it. Conjecture 3.9 below says that 1.13(ii) holds with $K(n)$ replaced by $Y(n)$. Computing $\pi_*(Y(n))$ is the main object of this paper. Each $Y(n)$ is a module over a spectrum

T_∞ (4.3) and we suspect (4.4) that T_∞ has the same Bousfield class as the sphere spectrum.

The functors L_n could be called chromatic localizations. There are natural transformations from L_{n+1} to L_n, so for each spectrum X we have an inverse system

$$L_0X \longleftarrow L_1X \longleftarrow L_2X \longleftarrow \cdots ,$$

and we can ask if the natural map from X to the homotopy inverse limit is an equivalence. This is the *chromatic convergence question*. The chromatic convergence theorem of Hopkins and the author [Rav92a, 7.5.7] says that this is the case for p-local finite spectrum X.

The *telescopic convergence question* concerns the inverse limit of the L_n^fX, its telescopic localizations. We know that there are maps

$$X \longrightarrow L_n^fX \longrightarrow L_nX$$

and that $\operatorname{holim} L_nX \simeq X$, so X is a retract of $\operatorname{holim} L_n^fX$. It suffices to answer this question for the case $X = S^0$, since $L_n^fX = X \wedge L_n^fS^0$ (L_n^f is smashing) and smashing with a finite complex preserves inverse limits.

2. Some variants of the Adams spectral sequence

The Adams spectral sequence for $\pi_*(X)$ is derived from the following *Adams diagram*.

$$\begin{array}{ccccccc} X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots \\ \big\downarrow{\scriptstyle g_0} & & \big\downarrow{\scriptstyle g_1} & & \big\downarrow{\scriptstyle g_2} & & \\ K_0 & & K_1 & & K_2 & & \end{array} \tag{2.1}$$

Here X_{s+1} is the fiber of g_s. We get an exact couple of homotopy groups and a spectral sequence with

$$E_1^{s,t} = \pi_{t-s}(K_s) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

This spectral sequence converges to $\pi_*(X)$ if the homotopy inverse limit $\lim_{\leftarrow} X_s$ is contractible and certain $\lim^1$ groups vanish. When X is connective, the Adams spectral sequenceis generally displayed like a first quadrant spectral sequence. For more background, see [Rav86].

Now suppose we have a generalized homology theory represented by a ring spectrum E. Then the *canonical E-based Adams resolution for X* is the diagram (2.1) with $K_s = E \wedge X_s$. More generally an *E-based Adams resolution for X* is such a diagram where K_s is such that the map $g_s \wedge E$ is the inclusion of a retract. Under certain hypotheses on E the resulting E_2-term is independent of the choice of resolution and can be identified as an Ext group. The classical Adams spectral sequence is the case where $E = H/p$, the mod p Eilenberg-Mac Lane spectrum, and the Adams-Novikov spectral sequence is the case where $E = BP$, the Brown-Peterson spectrum. We will have occasion to use a noncanonical Adams resolution below for a case where $E = H/p$. Then the condition on the diagram is that $H_*(g_s)$ be monomorphic for each s.

2.1. **The classical Adams spectral sequence.** Here we have

$$E_2^{s,t} = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X)),$$

where A_* is the dual Steenrod algebra, $H_*(X)$ is the mod p homology of X, and Ext is taken in the category of A_*-comodules. This group is the same as $\mathrm{Ext}_A(H^*(X), \mathbf{Z}/(p))$, where $H^*(X)$ is regarded as a module over the Steenrod algebra A. This group is not easy to compute in most cases. There is not a single nontrivial example where X is finite and this group is completely known, although there are good algorithms for computing it in low dimensions.

We recall the structure of A_*. When working over a field k we will use the notation $P(x)$ and $E(x)$ to denote polynomial and exterior algebras over k on x. As an algebra we have

$$A_* = \begin{cases} P(\xi_1, \xi_2, \ldots) & \text{with } |\xi_i| = 2^i - 1 \\ & \quad \text{for } p = 2 \\ P(\xi_1, \xi_2, \ldots) \otimes E(\tau_0, \tau_1, \ldots) & \text{with } |\xi_i| = 2p^i - 2 \\ & \quad \text{and } |\tau_i| = 2p^i - 1 \\ & \quad \text{for } p > 2. \end{cases}$$

For odd primes we will denote the polynomial and exterior factors by P_* and Q_* respectively. For $p = 2$, P_* and Q_* will denote $P(\xi_i^2)$ and $E(\xi_i)$ respectively. The coproduct is given by

$$\begin{aligned} \Delta(\xi_i) &= \sum_{0 \le j \le i} \xi_{i-j}^{p^j} \otimes \xi_j \qquad \text{where } \xi_0 = 1. \\ \text{and} \qquad \Delta(\tau_i) &= \tau_i \otimes 1 + \sum_{0 \le j \le i} \xi_{i-j}^{p^j} \otimes \tau_j. \end{aligned}$$

In §2.5 we will review some facts about Ext groups over Hopf algebras such as A_*, which we will refer to here when needed.

In §3 we will construct a spectrum $y(n)$ with

$$H_*(y(n)) = \begin{cases} P(\xi_1, \ldots, \xi_n) & \text{for } p = 2 \\ P(\xi_1, \ldots, \xi_n) \otimes E(\tau_0, \ldots, \tau_{n-1}) & \text{for } p > 2. \end{cases}$$

Let

$$B(n)_* = \begin{cases} A_*/(\xi_1, \dots, \xi_n) & \text{for } p = 2 \\ A_*/(\xi_1, \dots, \xi_n, \tau_0, \dots, \tau_{n-1}) & \text{for } p > 2 \end{cases} \tag{2.2}$$

Then we have $H_*(y(n)) = A_* \square_{B(n)_*} \mathbf{Z}/(p)$, and we can use the change-of-rings isomorphism (2.30) to prove

Proposition 2.3. *With notation as above*

$$Ext_{A_*}(\mathbf{Z}/(p), H_*(y(n))) = \mathrm{Ext}_{B(n)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)).$$

For future reference we record some information about this Ext group. For a fixed value of n let

$$P'_* \;=\; \begin{cases} P(\xi_{n+1}, \xi_{n+2}, \dots) & \text{for } p > 2 \\ P(\xi_{n+1}^2, \xi_{n+2}^2, \dots) & \text{for } p = 2 \end{cases} \tag{2.4}$$

$$Q'_* \;=\; \begin{cases} E(\tau_n, \tau_{n+1}, \dots) & \text{for } p > 2 \\ E(\xi_{n+1}, \xi_{n+2}, \dots) & \text{for } p = 2 \end{cases} \tag{2.5}$$

Then we have a Hopf algebra extension (2.31)

$$P'_* \longrightarrow B(n)_* \longrightarrow Q'_* \tag{2.6}$$

and a Cartan-Eilenberg spectral sequence (2.32) converging to the group of 2.3 with

$$\begin{aligned} E_2 &= \mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), \mathrm{Ext}_{Q'_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))) \\ &= \mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), V'). \end{aligned}$$

where

$$V' = P(v_n, v_{n+1}, \dots). \tag{2.7}$$

The structure of V' as a comodule over P' is given in 2.15. The elements v_{n+k} for $0 \leq k \leq n$ are permanent cycles. In §2.3 we will consider the effect of inverting v_n.

2.2. **The Adams-Novikov spectral sequence.** Here we have

$$E_2^{s,t} = \mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(X)).$$

Here we are taking Ext in the category of comodules over the Hopf *algebroid* $BP_*(BP)$. The difficulty of computing this group is comparable to the classical case.

The structure of $BP_*(BP)$ is as follows. As algebras we have

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \qquad \text{with } |t_i| = 2p^i - 2.$$

It is not a Hopf algebra (i.e., a cogroup object in the category of algebras), but a Hopf algebroid, which is a cogroupoid object in the category of algebras. (For more discussion of this definition see [Rav86, A1.1] or [Rav92a, B.3].) This means that in addition to a coproduct map Δ there is a right unit map $\eta_R : BP_* \to BP_*(BP)$. The formulas for these maps involve the formal group law and are somewhat complicated. We will give approximations for them now. Let

$$I = (p, v_1, v_2, \ldots) \subset BP_*.$$

Then we have

$$\begin{aligned} \Delta(t_i) &\equiv \sum_j t_j \otimes t_{i-j}^{p^j} \mod I \quad \text{where } t_0 = 1 \\ \text{and} \qquad \eta_R(v_i) &\equiv \sum_j v_j t_{i-j}^{p^j} \mod I^2 \quad \text{where } v_0 = p. \end{aligned}$$

There is an analog of (2.30) for Hopf algebroids stated as A1.3.12 in [Rav86]. We have

$$BP_*(y(n)) = BP_*/I_n[t_1, \ldots, t_n]. \tag{2.8}$$

The analog of 2.3 is the following.

Corollary 2.9.

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(y(n))) = \mathrm{Ext}_{BP_*(BP)/(t_1,\ldots,t_n)}(BP_*, BP_*/I_n).$$

When X is a finite complex of type n, the Adams-Novikov E_2-term for $\widehat{X}$ is surprisingly easy to compute. In some cases we can get a complete description of it, quite unlike the situation for X itself. *It was this computability that originally motivated the second author's interest in this problem.* For such X we know that $BP_*(\widehat{X}) = BP_*(L_nX) = v_n^{-1}BP_*(X)$, and $BP_*(X)$ is always annihilated by some power of the ideal

$$I_n = (p, v_1, \ldots, v_{n-1}) \subset BP_*.$$

More generally if X is a connective spectrum in which each element of $BP_*(X)$ is annihilated by some power of I_n, we have $BP_*(L_nX) = v_n^{-1}BP_*(X)$. The results of [Rav87] and the smash product theorem [Rav92a, 7.5.6] imply that the Adams-Novikov spectral sequence for L_nX converges to $\pi_*(L_nX)$.

Now assume for simplicity that $BP_*(X)$ is annihilated by I_n itself; this condition is satisfied in all of the examples we shall study here. This means that $v_n^{-1}BP_*(X)$ is a comodule over $v_n^{-1}BP_*(BP)/I_n$, which turns out to be much more manageable than $BP_*(BP)$ itself. There is a change-of-rings isomorphism (originally conceived by Morava and proved in the form we'll use in [MR77]) that enables us to replace $v_n^{-1}BP_*(BP)/I_n$ with a smaller Hopf algebra $\Sigma(n)$, which we now describe. Let

$$K(n)_* = \begin{cases} \mathbf{Q} & \text{for } n = 0 \\ \mathbf{Z}/(p)[v_n, v_n^{-1}] & \text{for } n > 0, \end{cases}$$

(this is the coefficient ring for Morava K-theory) and define a BP_*-module structure on it by sending v_m to zero for $m \neq n$. $K(n)_*$ for $n > 0$ is a graded field in the sense that every graded module over it is free. Then let

$$\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*,$$

where the tensor product on the right is with respect to the BP_*-module structure on $BP_*(BP)$ induced by the right unit map η_R. Using more precise information about η_R, we get the following explicit description of $\Sigma(n)$ as an algebra.

$$\Sigma(n) = K(n)_*[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i).$$

It is a Hopf algebra with coproduct inherited from that on $BP_*(BP)$. For a $BP_*(BP)$-comodule M, $K(n)_* \otimes_{BP_*} M$ is a comodule over $\Sigma(n)$.

Now we can state the change-of-rings theorem of [MR77].

Theorem 2.10. *Let M be a $BP_*(BP)$-comodule that is annihilated by the ideal I_n. Then there is a natural isomorphism*

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}M) = \operatorname{Ext}_{\Sigma(n)}(K(n)_*, K(n)_* \otimes_{BP_*} M).$$

The Ext group on the right is explicitly computable in many interesting cases. It is related to the continuous mod p cohomology of the strict automorphism group of the height n formal group law. This connection was first perceived by Morava and is explained in [Rav86, Chapter 6]. The methods given there lead to the following analog of 2.9.

Corollary 2.11. *With $BP_*(y(n))$ as in (2.8),*

$$\begin{aligned} &\operatorname{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*(y(n))) \\ &\quad = \operatorname{Ext}_{\Sigma(n)/(t_1,\ldots,t_n)}(K(n)_*, K(n)_*[v_n t_1^{p^n} - v_n^p t_1, \ldots, v_n t_n^{p^n} - v_n^{p^n} t_n]) \\ &\quad = P(v_{n+1}, \ldots, v_{2n}) \otimes \operatorname{Ext}_{\Sigma(n)/(t_1,\ldots,t_n)}(K(n)_*, K(n)_*) \\ &\quad = K(n)_*[v_{n+1}, \ldots, v_{2n}] \otimes E(h_{n+i,j} \colon 1 \le i \le n, 0 \le j \le n-1), \end{aligned}$$

where $h_{n+i,j} \in \operatorname{Ext}^{1,2p^j(p^{n+i}-1)}$ corresponds to the primitive element

$$t_{n+i}^{p^j} \in \Sigma(n)/(t_1, \ldots, t_n).$$

2.3. **The localized Adams spectral sequence.** The classical Adams spectral sequence is useless for studying the telescope $\widehat{X}$ because its homology is trivial. We need to replace it with the localized Adams spectral sequence; the first published account of it is due to Miller [Mil81]. It is derived from the Adams spectral sequence in the following way. The telescope $\widehat{X}$ is obtained from X by iterating a v_n-map $f : X \to \Sigma^{-d}X$. Suppose there is a lifting

$$\tilde{f} : X \to \Sigma^{-d}X_{s_0}$$

(where X_{s_0} is as in (2.1)) for some $s_0 \geq 0$. This will induce maps $\tilde{f} : X_s \to \Sigma^{-d}X_{s+s_0}$ for $s \geq 0$. This enables us to define $\widehat{X}_s$ to be the homotopy direct limit of

$$X_s \xrightarrow{\quad \tilde{f} \quad} \Sigma^{-d}X_{s+s_0} \xrightarrow{\quad \tilde{f} \quad} \Sigma^{-2d}X_{s+2s_0} \xrightarrow{\quad \tilde{f} \quad} \cdots$$

Let $X_s = X$ for $s < 0$. Thus we get the following diagram, similar to that of (2.1).

$$\begin{array}{ccccccccc} \cdots \longleftarrow & \widehat{X}_{-1} & \longleftarrow & \widehat{X}_0 & \longleftarrow & \widehat{X}_1 & \longleftarrow & \cdots \\ & \Big\downarrow g_{-1} & & \Big\downarrow g_0 & & \Big\downarrow g_1 & & \\ & \widehat{K}_{-1} & & \widehat{K}_0 & & \widehat{K}_1, & & \end{array} \tag{2.12}$$

where the spectra $\widehat{K}_s$ are defined after the fact as the obvious cofibers. This leads to a full plane spectral sequence (the localized Adams spectral sequence) with

$$E_1^{s,t} = \pi_{t-s}(\widehat{K}_s) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

as before. This spectral sequence converges to the homotopy of the homotopy direct limit $\pi_*(\lim_{\to} \widehat{X}_{-s})$ if the homotopy inverse limit $\lim_{\leftarrow} \widehat{X}_s$ is contractible.

Theorem 2.13 (Convergence of the localized Adams spectral sequence)**.** *For a spectrum X equipped with maps f and $\tilde{f}$ as above, in the localized Adams spectral sequence for $\pi_*(\widehat{X})$ we have*

- *The homotopy direct limit* $\lim_{\to} \widehat{X}_{-s}$ *is the telescope* $\widehat{X}$*.*
- *The homotopy inverse limit* $\lim_{\leftarrow} \widehat{X}_s$ *is contractible if the original (unlocalized) Adams spectral sequence has a vanishing line of slope s_0/d at E_r for some finite r, i.e., if there are constants c and r such that*

$$E_r^{s,t} = 0 \quad \text{for} \quad s > c + (t-s)(s_0/d).$$

(In this case we say that f has a parallel lifting $\tilde{f}$*.)*

- *If f has a parallel lifting, this localized Adams spectral sequence converges to $\pi_*(\widehat{X})$.*

Proof. For the assertion about the homotopy direct limit, note that

$$\begin{aligned} \widehat{X}_s &= \lim_{\to i} \Sigma^{-di} X_{s+is_0} \\ \text{so} \qquad \lim_{\to s} \widehat{X}_s &= \lim_{\to s} \lim_{\to i} \Sigma^{-di} X_{s+is_0} \\ &= \lim_{\to i} \lim_{\to s} \Sigma^{-di} X_{s+is_0} \\ &= \lim_{\to i} \Sigma^{-di} X \\ &= \widehat{X}. \end{aligned}$$

Next we will prove the assertion about the vanishing line. Let $E_r^{s,t}(X)$ denote the E_r-term of the Adams spectral sequence for X, and $E_r^{s,t}(\widehat{X})$ that of the localized Adams spectral sequence. Then $\tilde{f}$ induces homomorphisms

$$E_r^{s,t}(X) \xrightarrow{\tilde{f}} E_r^{s+s_0,t}(\Sigma^{-d}X) = E_r^{s+s_0,t+d}(X)$$

and we have

$$E_r^{s,t}(\widehat{X}) = \lim_{\to k} E_r^{s+ks_0,t+kd}(X),$$

so the vanishing line of the localized Adams spectral sequence follows from that of the unlocalized Adams spectral sequence.

Next we will show that $\lim_{\leftarrow}(\hat{X}_i)$ is contractible. Recall that

$$\hat{X}_i = \lim_{\to k} \Sigma^{-kd} X_{i+ks_0}$$

so

$$\pi_m(\hat{X}_i) = \lim_{\to k} \pi_{m+kd}(X_{i+ks_0}).$$

Now the vanishing line implies that the map $g : X_s \to X_{s-r+1}$ satisfies $\pi_m(g) = 0$ for $m < (sd + c)/s_0$. To see this, note that a permanent cycle of filtration s corresponds to a coset (modulo the image of $\pi_*(X_{s+1})$) in $\pi_*(X_s)$. It is dead in the E_r-term if and only if its image in $\pi_*(X_{s-r+1})$ is trivial.

It follows that for each $k > 0$ we have a diagram

$$\begin{array}{ccc} X_s & \xrightarrow{g} & X_{s-r+1} \\ \downarrow & & \downarrow \\ \Sigma^{-dk} X_{s+s_0k} & \xrightarrow{g} & \Sigma^{-dk} X_{s+s_0k-r+1} \end{array}$$

in which both maps g vanish on π_m for $m < (sd+c)/s_0$. Hence the map

$$\widehat{X}_s \xrightarrow{\hat{g}} \widehat{X}_{s-r+1}$$

has the same property.

It follows that if we fix m and s, the homomorphism

$$\pi_m(\hat{X}_i) \longrightarrow \pi_m(\hat{X}_s) \tag{2.14}$$

is trivial for sufficiently large i, and the image of

$$\lim_{\leftarrow} \pi_*(\hat{X}_i) \longrightarrow \pi_*(\hat{X}_s)$$

is trivial for each s, so

$$\lim_{\leftarrow} \pi_*(\hat{X}_i) = 0.$$

To complete the proof that $\lim_{\leftarrow}(\hat{X}_i)$ is contractible, we need to show that

$$\lim_{\leftarrow}{}^1 \pi_*(\hat{X}_i) = 0.$$

However, (2.14) implies that the inverse system of homotopy groups is Mittag-Leffler, so $\lim^1_{\leftarrow}$ vanishes.

According to Boardman [Boa81, §10], the convergence of a whole plane spectral sequence such as ours requires, in addition to the contractibility just proved, the vanishing of a certain obstruction group that he calls W. (It measures the failure of certain direct and inverse limits to commute.) However, his Lemma 10.3 says that our vanishing line implies that $W = 0$. □

Here are some informative examples.

- If we start with the Adams-Novikov spectral sequence, then the map f cannot be lifted since $BP_*(f)$ is nontrivial. Thus we have $s_0 = 0$ and the lifting condition requires that X has a horizontal vanishing line in its Adams-Novikov spectral sequence. This is not known (or suspected) to occur for any nontrivial finite X, so we do not get a convergence theorem about the localized Adams-Novikov spectral sequence, which is merely the standard Adams-Novikov spectral sequence applied to $\widehat{X}$.
- If we start with the classical Adams spectral sequence, theorem of Hopkins-Palmieri-Smith says that a type n X (with $n > 0$) always has a vanishing line of slope $1/|v_n| = 1/(2p^n - 2)$, in the E_r term, for some r. ([HPS98]) Thus we have convergence if f has a lifting with $s_0 = d/|v_n|$. This does happen in the few

cases where Toda's complex $V(n-1)$ exists. Then $V(n-1)$ is a type n complex with a v_n-map with $d = |v_n|$ and $s_0 = 1$.
- In favorable cases (such as Toda's examples and $y(n)$) the E_2-term of the localized Adams spectral sequence can be identified as an Ext groups which can be computed explicitly.

We will discuss the last example in more detail. *For simplicity we assume until further notice that p is odd.* Recall from §2.1 that

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(y(n))) = \mathrm{Ext}_{B(n)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

and that the latter can be computed using Cartan-Eilenberg spectral sequence (2.32) for the extension (2.6) with

$$E_2 = \mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), V').$$

The effect of localization is to invert v_n as in (2.35). The comodule structure on V' is given by

$$\begin{aligned}\psi(v_{2n+i}) &= 1 \otimes v_{2n+i} + \sum_{0 \le k < i} \xi_{n+i-k}^{p^{n+k}} \otimes v_{n+k} \\ &= 1 \otimes v_{2n+i} + \xi_{n+i}^{p^n} \otimes v_n + \dots .\end{aligned}$$

In the ring $v_n^{-1}V'$, define w_{2n+i} for $i > 0$ recursively by

$$w_{2n+i} = v_n^{-1} \left(v_{2n+i} - \sum_{0<k<i} v_{n+k} w_{2n+i-k}^{p^k} \right), \tag{2.15}$$

and let

$$W' = P(v_n, v_{n+1}, ..., v_{2n}, w_{2n+1}, w_{2n+2}, \dots) \subset v_n^{-1}V'. \tag{2.16}$$

(2.15) can be rewritten as

$$v_{2n+i} = \sum_{0 \le k < i-n} v_{n+k} w_{2n+i-k}^{p^k}. \tag{2.17}$$

We will show below (see (5.7)) that for w_{2n+i} as in (2.15),

$$\psi(w_{2n+i}) = 1 \otimes w_{2n+i} - \overline{\xi}_{n+i}^{p^n} \otimes 1 + \sum_{n<k<i} \overline{\xi}_k^{p^n} \otimes w_{2n+i-k}^{p^k}. \tag{2.18}$$

Then using (2.28) we have

$$d(w_{2n+i}) = -\overline{\xi}_{n+i}^{p^n} \otimes 1 + \sum_{n<k<i} \overline{\xi}_k^{p^n} \otimes w_{2n+i-k}^{p^k}.$$

in the cobar complex $C_{P'_*}(W')$. Hence the expression on the right is a cocycle, so the same is true of its p^{j-n}th power in the algebra $P'_*\otimes W'$, (for $j<n$, not $j\geq n$, as one might expect,)

$$h_{n+i,j} = -\overline{\xi}_{n+i}^{p^j}\otimes 1 + \sum_{n<k<i}\overline{\xi}_k^{p^j}\otimes w_{2n+i-k}^{p^{j+k-n}}, \tag{2.19}$$

and we denote the element in Ext represented by its transpotent by $b_{n+i,j}$.

It follows that W' (2.16) and therefore $v_n^{-1}V'$ are free comodules over

$$P(\xi_{n+1}^{p^n}, \xi_{n+2}^{p^n}, \ldots).$$

Using the localized change-of-rings isomorphism of (2.33), we get

$$\begin{aligned} & v_n^{-1}\mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), V') \\ & \quad = \mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), v_n^{-1}V') \\ & \quad = \mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), v_n^{-1}W') \\ & \quad = v_n^{-1}\mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), W') \\ & \quad = v_n^{-1}P(v_n, \ldots, v_{2n})\otimes \mathrm{Ext}_{P'_*/(\xi_{n+i}^{p^n})}(\mathbf{Z}/(p), \mathbf{Z}/(p)). \end{aligned}$$

The Ext group above is easy to compute because the coproduct in $P'_*/(\xi_{n+i}^{p^n})$ is trivial, i.e., each generator is primitive. Thus we have

$$\begin{aligned} & v_n^{-1}\mathrm{Ext}_{P'_*}(\mathbf{Z}/(p), V') \\ & = v_n^{-1}P(v_n, \ldots, v_{2n})\otimes \begin{cases} E(h_{n+i,j} : i>0, 0\leq j<n) \\ \quad \otimes P(b_{n+i,j} : i>0, 0\leq j<n) \\ \qquad \text{for } p \text{ odd} \\ P(h_{n+i,j} : i>0, 0\leq j<n) \\ \qquad \text{for } p=2. \end{cases} \end{aligned} \tag{2.20}$$

Since the elements v_{n+i} for $0\leq i\leq n$ are permanent cycles, the Cartan-Eilenberg spectral sequence collapses. There are no multiplicative extensions since E_∞ has no zero divisors. Hence the above is a description of the E_2-term of the localized Adams spectral sequence for $Y(n)$.

2.4. **The Thomified Eilenberg-Moore spectral sequence.** We will use a Thomified form of the Eilenberg-Moore spectral sequence which is introduced in [MRS].

Let

$$X \xrightarrow{i} E \xrightarrow{h} B \tag{2.21}$$

be a fiber sequence with simply connected base space B, and suppose that we also have a p-local stable spherical fibration ξ over E which is oriented with respect to mod p homology.

Let Y, and K be the Thomifications of X and E. In [MRS] we construct a diagram

$$\begin{array}{ccccccc} Y & = & Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots \\ & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\ & & K_0 & & K_1 & & K_2 & & \end{array} \tag{2.22}$$

where Y_{s+1} is the fiber of g_s and

$$H_*(K_s) = \Sigma^{-s} H_*(K) \otimes \overline{H}_*(B^{(s)}),$$

for $s > 0$. This is similar to the Adams diagram of (2.1), but $H_*(g_s)$ need not be a monomorphism in general. As before the associated exact couple of homotopy groups leads to a spectral sequence, which we will call the *Thomified Eilenberg-Moore spectral sequence.*

To identify the E_2-term in certain cases, note that $H_*(K)$ is simultaneously a comodule over A_* and (via the Thom isomorphism and the map h_*) $H_*(B)$, which is itself a comodule over A_*. Following Massey-Peterson [MP67], we combine these two structures by defining the *semitensor product* coalgebra

$$R_* = H_*(B) \otimes A_* \tag{2.23}$$

in which the coproduct is the composite

$$\begin{array}{c} H_*(B)\otimes A_* \\ \Big\downarrow{\scriptstyle \Delta_B\otimes\Delta_A} \\ H_*(B)\otimes H_*(B)\otimes A_*\otimes A_* \\ \Big\downarrow{\scriptstyle H_*(B)\otimes\psi_B\otimes A_*\otimes A_*} \\ H_*(B)\otimes A_*\otimes H_*(B)\otimes A_*\otimes A_* \\ \Big\downarrow{\scriptstyle H_*(B)\otimes A_*\otimes T\otimes A_*} \\ H_*(B)\otimes A_*\otimes A_*\otimes H_*(B)\otimes A_* \\ \Big\downarrow{\scriptstyle H_*(B)\otimes m_A\otimes H_*(B)\otimes A_*} \\ (H_*(B)\otimes A_*)\otimes (H_*(B)\otimes A_*), \end{array} \tag{2.24}$$

where Δ_A and Δ_B are the coproducts on A_* and $H_*(B)$, T is the switching map, $\psi_B : H_*(B) \to A_* \otimes H_*(B)$ is the comodule structure map, and m_A is the multiplication in A_*.

Massey-Peterson gave this definition in cohomological terms. They denoted the semitensor algebra R by $H^*(B) \odot A$, which is additively isomorphic to $H^*(B) \otimes A$ with multiplication given by

$$(x_1 \otimes a_1)(x_2 \otimes a_2) = x_1 a_1'(x_2) \otimes a_1'' a_2,$$

where $x_i \in H^*(B)$, $a_i \in A$, and $a_1' \otimes a_1''$ denotes the coproduct expansion of a_1 given by the Cartan formula. Our definition is the homological reformulation of theirs.

Note that given a map $f : V \to B$ and a subspace $U \subset V$, $\bar{H}^*(V/U) = H^*(V, U)$ is an R-module since it is an $H^*(V)$-module via relative cup products, even if the map f does not extend to the quotient V/U. In our case we have maps $G_s \to B$ for all $s \geq 0$ given by

$$(e, b_1, \ldots, b_s) \mapsto h_e,$$

induced form the standard form of the Eilenberg-Moore spectral sequence, as in [MRS]. These are compatible with all of the maps h_t, so $H_*(Y_s)$ and $H_*(K_s)$ are R_*-comodules, and the maps between them respect this structure.

We will see in the next theorem that under suitable hypotheses, the E_2-term of the Thomified Eilenberg-Moore spectral sequence is

$\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K))$ when B is an H-space. When B is an H-space we have a Hopf algebra extension (2.31)

$$A_* \longrightarrow R_* \longrightarrow H_*(B).$$

This gives us a Cartan-Eilenberg spectral sequence (2.32) converging to this Ext group with

$$E_2 = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), \mathrm{Ext}_{H_*(B)}(\mathbf{Z}/(p), H_*(K))). \tag{2.25}$$

Note that the inner Ext group above is the same as $\mathrm{Ext}_{H_*(B)}(\mathbf{Z}/(p), H_*(E))$, the E_2-term of the classical Eilenberg-Moore spectral sequence converging to $H_*(X)$. If the latter collapses from E_2 (which it does in the examples we will study), then the Ext group of (2.25) can be thought of as

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(Y)),$$

where $H_*(Y)$ is equipped with the Eilenberg-Moore bigrading. This is the usual Adams E_2-term for Y when $H_*(Y)$ is concentrated in Eilenberg-Moore degree 0, but not in general.

Theorem 2.26. (i) *Suppose that $H^*(K)$ is a free A-module and B is simply connected. Then the Thomified Eilenberg-Moore spectral sequence associated with the homotopy of (2.22) converges to $\pi_*(Y)$ with*

$$E_2 = \mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K)),$$

where R_* is the Massey-Peterson coalgebra of (2.23).

(ii) *If in addition the map $i : X \to E$ induces a monomorphism in mod p homology, then the Thomified Eilenberg-Moore spectral sequence coincides with the classical Adams spectral sequence for Y.*

This is proved in [MRS]. Now we give a corollary that indicates that the hypotheses are not as restrictive as they may appear.

Corollary 2.27. *Given a fibration*

$$X \longrightarrow E \longrightarrow B$$

with X p-adically complete, a p-local spherical fibration over E, and B simply connected, there is a spectral sequence converging to $\pi_*(Y)$ (where Y is the Thomification of X) with

$$E_2 = \mathrm{Ext}_{H_*(B)\otimes A_*}(\mathbf{Z}/(p), H_*(K)),$$

where K as usual is the Thomification of E.

Proof. We can apply 2.26 to the product of the given fibration with pt. $\to \Omega^2 S^3 \to \Omega^2 S^3$, where $\Omega^2 S^3$ is equipped with the p-local spherical fibration of Lemma 3.3 below. Then the Thomified total space is $K \wedge H/p$, so its cohomology is a free A-module. Thus the E_2-term is

$$\mathrm{Ext}_{H_*(B\wedge H/p)\otimes A_*}(\mathbf{Z}/(p), H_*(K\wedge H/p)) = \mathrm{Ext}_{H_*(B)\otimes A_*}(\mathbf{Z}/(p), H_*(K)).$$

□

2.5. **Hopf algebras and localized Ext groups.** In this subsection we will collect some results about Ext groups over Hopf algebras and their localizations. We refer the reader to [Rav86, A1.3] for details of the unlocalized theory.

Given a connected graded cocommutative Hopf algebra Γ over a field k (always $\mathbf{Z}/(p)$ in this paper) and a left Γ-comodule M, there is a cobar complex $C_\Gamma(M)$ whose cohomology is $\mathrm{Ext}_\Gamma(k, M)$; see [Rav86, A1.2.11] where it is denoted by $C_\Gamma(k, M)$. Additively we have

$$C^s_\Gamma(M) = \Gamma^{\otimes s} \otimes M.$$

The coboundary on $C^0_\Gamma(M) = M$ is given by

$$d(m) = \psi(m) - 1 \otimes m \tag{2.28}$$

where $\psi : M \to \Gamma \otimes M$ is the comodule structure map. When M is a comodule algebra, $C_\Gamma(M)$ is a differential graded algebra. The product is somewhat complicated and is given in [Rav86, A1.2.15]. For future reference we record the formula for

$$C^1_\Gamma(M) \otimes C^1_\Gamma(M) \xrightarrow{\cup} C^2_\Gamma(M),$$

namely

$$(\gamma_1 \otimes m_1) \cup (\gamma_2 \otimes m_2) = \pm \gamma_1 \otimes m_1' \gamma_2 \otimes m_1'' m_2, \tag{2.29}$$

where $m_1' \otimes m_1''$ denotes the comodule expansion of m_1.

Given a Hopf algebra map $f : \Gamma \to \Phi$ and a left Φ-comodule M, there is a spectral sequence converging to $\mathrm{Ext}_\Phi(k, M)$ with

$$E_2^{i,j} = \mathrm{Ext}_\Gamma(k, \mathrm{Ext}_\Phi(k, \Gamma \otimes M)) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}. \tag{2.29}$$

It is derived from the double complex $C_\Gamma(C_\Phi(\Gamma \otimes M))$ by filtering by the first degree. More explicitly we have

$$C^i_\Gamma(C^j_\Phi(\Gamma \otimes M)) = \Gamma^{\otimes i} \otimes \Phi^{\otimes j} \otimes \Gamma \otimes M$$

The jth row is $C_\Gamma(\Phi^{\otimes j} \otimes \Gamma \otimes M)$, which is acyclic since the comodule $\Phi^{\otimes j} \otimes \Gamma \otimes M$ is free over Γ. This means that filtering by the second degree and computing the cohomology of each row first gives us $C_\Phi(M)$ in the 0th column. This shows that the total complex is chain homotopy equivalent to $C_\Phi(M)$ and its cohomology is $\mathrm{Ext}_\Phi(k, M)$.

On the other hand, the ith column is

$$\Gamma^{\otimes i} \otimes C_\Phi(\Gamma \otimes M)$$

so its cohomology is

$$\Gamma^{\otimes i} \otimes \mathrm{Ext}_\Phi(k, \Gamma \otimes M)$$

giving

$$E_1^{i,j} = C_\Gamma^i(\mathrm{Ext}_\Phi^j(k, \Gamma \otimes M))$$

and

$$E_2^{i,j} = \mathrm{Ext}_\Gamma^i(k, \mathrm{Ext}_\Phi^j(k, \Gamma \otimes M))$$

as claimed.

There are two interesting cases of this spectral sequence, occurring when f is surjective and when it is injective. When it is surjective the inner Ext group is $\Gamma \Box_\Phi M$ concentrated in degree 0 since Γ is a free Φ-comodule. Hence the spectral sequence collapses and we have

$$\mathrm{Ext}_\Phi(k, M) = \mathrm{Ext}_\Gamma(k, \Gamma \Box_\Phi M). \tag{2.30}$$

This is the *change-of-rings isomorphism* due originally to Milnor-Moore [MM65].

The other interesting case of the spectral sequence occurs when we have an extension of Hopf algebras

$$\Gamma \xrightarrow{f} \Phi \xrightarrow{g} \Lambda; \tag{2.31}$$

this means that $\Phi = \Gamma \otimes \Lambda$ both as Γ-modules and as Λ-comodules. Applying (2.30) to the surjection g gives

$$\mathrm{Ext}_\Phi(k, \Gamma \otimes M) = \mathrm{Ext}_\Phi(k, \Sigma \Box_\Lambda M) = \mathrm{Ext}_\Lambda(k, M)$$

so the E_2-term of the spectral sequence associated with f is

$$E_2^{i,j} = \mathrm{Ext}_\Gamma^i(k, \mathrm{Ext}_\Lambda^j(k, M)). \tag{2.32}$$

This is the Cartan-Eilenberg spectral sequence of [CE56, page 349].

Now we will discuss localized Ext groups. Suppose a Hopf algebra Γ has an odd dimensional (this is not needed if k has characteristic 2) primitive element t. Then there is a corresponding element $v \in$

$\mathrm{Ext}^1_\Gamma(k,k)$ which we would like to invert. The class v is represented by a short exact sequence

$$0 \longrightarrow k \longrightarrow L \longrightarrow \Sigma^{|t|}k \longrightarrow 0$$

of Γ-comodules. Now suppose we have an injective Γ-resolution (such as the one associated with the cobar complex or the double complex above) of a left Γ-comodule M,

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots$$

and let $J_s = \ker d_s = \mathrm{coker} d_{s-1}$. Then for each $s \geq 0$ we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_s & \longrightarrow & L \otimes J_s & \longrightarrow & \Sigma^{|t|}J_s & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_s & \longrightarrow & I_s & \longrightarrow & J_{s+1} & \longrightarrow & 0 \end{array}$$

Using this we get a diagram

$$\begin{array}{ccccccc} M & \longrightarrow & \Sigma^{-|t|}J_1 & \longrightarrow & \Sigma^{-2|t|}J_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ I_0 & \longrightarrow & \Sigma^{-|t|}I_1 & \longrightarrow & \Sigma^{-2|t|}I_2 & \longrightarrow & \cdots, \end{array}$$

(where the maps in the bottom row exist because their targets are injective and the vertical maps are inclusions) and hence a direct limit of injective resolutions, of the corresponding cochain complexes obtained by cotensoring over Γ with k, and of Ext groups. We denote the direct limit of cobar complexes by $v^{-1}C_\Gamma(M)$ (the *localized cobar complex*) and its cohomology by $v^{-1}\mathrm{Ext}_\Gamma(M)$, the *localized Ext group.*

Now suppose we have a map $f : \Gamma \to \Phi$ as before with an odd dimensional primitive $t \in \Gamma$ corresponding to $v \in \mathrm{Ext}^1_\Gamma(k, M)$. We can replace the double complex $C_\Gamma(C_\Phi(\Gamma \otimes M))$ by $v^{-1}C_\Gamma(C_\Phi(\Gamma \otimes M))$. The equivalence between $C_\Gamma(C_\Phi(\Gamma \otimes M))$ and $C_\Phi(M)$ is preserved by inverting v in this way, so we get a spectral sequence converging to $v^{-1}\mathrm{Ext}_\Phi(k, M)$. The ith column of the double complex is $v^{-1}C^i_\Gamma(C_\Sigma(\Gamma \otimes M))$, and we get

$$E_2^{i,j} = v^{-1}\mathrm{Ext}^i_\Gamma(k, \mathrm{Ext}^j_\Phi(k, M)).$$

When f is onto, the inner Ext group collapses as before and we get a localized change-of-rings isomorphism

$$v^{-1}\mathrm{Ext}_\Phi(k, M) = v^{-1}\mathrm{Ext}_\Gamma(k, \Gamma\Box_\Phi M), \tag{2.33}$$

and when f is the injection in a Hopf algebra extension as in (2.31) we get the first form of the localized Cartan-Eilenberg spectral sequence

$$v^{-1}\mathrm{Ext}_\Gamma(k, \mathrm{Ext}_\Lambda(k, M)) \implies v^{-1}\mathrm{Ext}_\Phi(k, M). \tag{2.34}$$

We can also consider the case where the odd dimensional primitive t is in Φ but not in Γ. Then we replace the double complex $C_\Gamma(C_\Phi(\Gamma \otimes M))$ by $C_\Gamma(v^{-1}C_\Phi(\Gamma \otimes M))$. Then again we have acyclic rows and taking their cohomology gives $v^{-1}C_\Phi(M)$ in the 0th column. Thus our spectral sequence converges again to $v^{-1}\mathrm{Ext}_\Phi(k, M)$ with

$$E_2^{i,j} = \mathrm{Ext}^i_\Gamma(k, v^{-1}\mathrm{Ext}^j_\Phi(k, M)).$$

In the case of an extension we use (2.33) to identify the inner Ext group, and we get the second form of the localized Cartan-Eilenberg spectral sequence

$$\mathrm{Ext}_\Gamma(k, v^{-1}\mathrm{Ext}_\Lambda(k, M)) \implies v^{-1}\mathrm{Ext}_\Phi(k, M). \tag{2.35}$$

3. The spectra $y(n)$ and $Y(n)$

We will now construct the spectrum $y(n)$ whose homology and E_2-terms were discussed previously, along with the associated telescope $Y(n)$.

3.1. **The EHP sequence and some Thom spectra.** Recall that ΩS^3 is homotopy equivalent to a CW-complex with a single cell in every even dimension. Let $J_m S^2$ (the mth James product of S^2) denote its $2m$-skeleton. James [Jam55] showed that there is a splitting

$$\Sigma\Omega S^3 \simeq \bigvee_{i>0} S^{2i+1}.$$

These lead to the James-Hopf maps $H_i\colon \Omega S^3 \to \Omega S^{2i+1}$ which are surjective in homology. We will denote H_p simply by H. When i is a power of a prime p, the p-local fiber of this map is a skeleton, i.e., there is a p-local fiber sequence

$$J_{p^n-1}S^2 \longrightarrow \Omega S^3 \longrightarrow \Omega S^{2p^n+1}. \tag{3.1}$$

Definition 3.2. *$y(n)$ is the Thom spectrum of the p-local spherical fibration over $\Omega J_{p^{n-1}} S^2$ induced from the one over $\Omega^2 S^3$ given by Lemma 3.3 below.*

$y(n)$ is an A_∞ ring spectrum, since it is the Thom spectrum of a bundle induced by a loop map ([Mah79].) It may be that in the cases where Toda's complex $V(n-1)$ exists and p is odd, that $y(n) \simeq V(n-1) \wedge T(n)$ (but probably not as A_∞ ring spectra), where $T(n)$ is the spectrum of [Rav86, §6.5] with

$$BP_*(T(n)) = BP_*[t_1, t_2, \ldots, t_n].$$

It is a p-local summand of the Thom spectrum of the canonical complex bundle over $\Omega SU(p^n)$.

The following is proved in [MRS].

Lemma 3.3. *For each prime p there is a p-local spherical fibration over $\Omega^2 S^3$ whose Thom spectrum is the mod p Eilenberg-Mac Lane spectrum H/p.*

For the rest of this section *we assume the p is odd* to avoid notational complications. We have

$$H_*(y(n)) = E(\tau_0, \tau_1, \cdots \tau_{n-1}) \otimes P(\xi_1, \cdots \xi_n)$$

as comodules over A_*, as can be inferred from [Mah79].

Lemma 3.4. *$y(n)$ is a split ring spectrum, i.e., $y(n) \wedge y(n)$ is a wedge of suspensions of $y(n)$ with one summand for each basis element of $H_*(y(n))$. In particular*

$$y(n)_*(y(n)) = \pi_*(y(n)) \otimes H_*(y(n)).$$

Proof. Consider the Atiyah-Hirzebruch spectral sequence for $y(n)_*(y(n))$ with

$$E_2 = H_*(y(n); \pi_*(y(n))).$$

It suffices to show that each multiplicative generator of $H_*(y(n))$ is a permanent cycle. These generators all have dimensions no more than $|v_n|$, and below that dimension $y(n)$ is equivalent to H/p. It follows that there are no differentials in the Atiyah-Hirzebruch spectral sequence in that range. □

The classical Adams E_2-term for $y(n)$ was described in Corollary 2.3. In low dimensions there is no room for any differentials, and we have

Lemma 3.5. *Below dimension $2p^{2n+1} - 2p^{n-1} - 2$, the Adams spectral sequence for $\pi_*(y(n))$ collapses from E_2 (for formal reasons), with*

$$E_2 = P(v_n, \cdots, v_{2n}) \otimes E(h_{n+i,j} : i > 0, j \geq 0) \otimes P(b_{n+i,j} : i > 0, j \geq 0),$$

where

$$\begin{aligned} v_{n+i} &\in E_2^{1,2p^{n+i}-1} \\ h_{n+i,j} &\in E_2^{1,2p^{n+i+j}-2p^j} \\ b_{n+i,j} &\in E_2^{2,2p^{n+i+j+1}-2p^{j+1}}. \end{aligned}$$

Proof. From the Hopf algebra extension

$$H_*(y(n)) \longrightarrow A_* \longrightarrow B(n)_*$$

we see that $\mathrm{Ext}_{B(n)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*y(n))$ (our E_2-term) is a comodule over $H_*(y(n))$, regarded as a subalgebra of A_*. From 3.4 we see that this subalgebra of A_* is part of the coalgebra of co-operations in $y(n)_*$-theory. This means that the corresponding quotient of A acts on the Adams spectral sequence. Routine calculations give

$$\begin{aligned} \beta(v_{n+i}) &= h_{n+i,0}, \\ P^{p^j}(h_{n+i,j}) &= h_{n+i-1,j+1} \\ \text{and} \qquad P^{p^{j+1}}(b_{n+i,j}) &= b_{n+i-1,j+1}. \end{aligned}$$

Hence if we can show that v_{n+i} for $i \leq n$ and $b_{n+i,0}$ for $i \leq n-1$ are permanent cycles, then the same will be true of all generators in our range of dimensions. We will show this by proving that there are no elements (besides $h_{n+i,0}$) in dimension $|v_{n+i}| - 1$ or $|b_{b+i,0}| - 1$ for these i.

This can be done by organizing the information in a suitable way. We define the *weight* $||x||$ of an element x by

$$\begin{aligned} ||v_{n+i}|| &= p^i, \\ ||h_{n+i,j}|| &= p^{i+j}, \\ ||b_{n+i,j}|| &= p^{i+j+1} \\ \text{and} \qquad ||xy|| &= ||x|| + ||y||. \end{aligned}$$

The generator x having the lowest dimension for its weight is

$$\begin{cases} v_n & \text{if } ||x|| = 1, \\ h_{n+1,0} & \text{if } ||x|| = p \qquad \text{and} \\ b_{n+1,j} & \text{if } ||x|| = p^{j+2} \quad \text{for } j \geq 0. \end{cases}$$

and the one with the highest weight is always v_{n+i}.

Next observe that for $i \leq n$,

$$\begin{aligned} (p^i + 1)|b_{n+1,j}| &> |v_{n+i+j+2}| \\ \text{and} \qquad (p^i - 1)|v_{n+k}| &< |b_{n+1,i+k-2}|. \end{aligned}$$

This means that in our range *the target of a differential on a generator x must have the same weight as x.*

In general, Adams spectral sequence differentials need not preserve this weight function. However, the first possible exceptions to this occur just outside our range, namely it is possible that

(3.6)
$$\begin{array}{rrcll} & d_1(v_{2n+1}) & = & v_n h_{n+1,n} & \\ \text{and} & d_{2p^j}(h_{2n+1-j,j}) & = & v_n b_{n+1,n-1-j}^{p^j} & \text{for } 0 \le j \le n-1. \end{array}$$

We will see below (5.18) that these differentials actually occur, the first being apparent from the structure of $B(n)_*$.

Now consider the quantity

$$\mu(x) = |x| - 2p^n ||x||,$$

which satisfies $\mu(xy) = \mu(x) + \mu(y)$. Then we have

$$\begin{array}{rrcl} & \mu(v_{n+i}) & = & -2, \\ & \mu(h_{n+i,j}) & = & -1 - 2p^j \\ \text{and} & \mu(b_{n+i,j-1}) & = & -2 - 2p^j. \end{array}$$

From this we can see that for any monomial x of weight p^i, $\mu(v_{n+i})$ exceeds $1 + \mu(x)$ except when $x = h_{n+i,0}$, and $\mu(b_{n+i-1,0})$ exceeds it except when x is one of the three generators with a higher value of μ, namely v_{n+i}, $h_{n+i,0}$ and $h_{n+i-1,1}$.

We know that $d_r(v_{n+i})$ must have weight p^i and that $\mu(d_r(v_{n+i})) = \mu(v_{n+i})-1$, so there is no possible nontrivial value of $d_r(v_{n+i})$. Similarly there can be no differential on $b_{n+i-1,0}$. □

The first positive dimensional element in $\pi_*(y(n))$ is $v_n \in \pi_{2p^n-2}(y(n))$. We can use the multiplication on $y(n)$ to extend v_n to a self-map. The telescope $Y(n)$ is the homotopy colimit of

$$y(n) \xrightarrow{v_n} \Sigma^{-|v_n|} y(n) \xrightarrow{v_n} \Sigma^{-2|v_n|} y(n) \xrightarrow{v_n} \cdots \tag{3.7}$$

Theorem 3.8. *The telescope $Y(n)$ defined above is $L_n^f y(n)$.*

Proof. We will adapt the methods used by Hopkins-Smith [HS98] to prove the periodicity theorem, as explained in [Rav92a, Chapter 6]. Let X by a finite complex of type n with a v_n-map f such that $K(n)_*(f)$ is multiplication by v_n^k; see [Rav92a, 6.1.1]. Let $R = DX \wedge X$, which is a finite ring spectrum. We will compute in $\pi_*(R \wedge y(n))$, which is a noncommutative $\mathbf{Z}/(p)$-algebra. Let $F \in \pi_*(R \wedge y(n))$ denote the image of f under map $R \to R \wedge y(n)$, and let G be the image of $g = v_n^k$ under the map $y(n) \to R \wedge y(n)$.

Now $R \wedge y(n)$ has an Adams vanishing line of slope $1/|v_n|$ since $y(n)$ does. The map $F - G$ represents an element above the line of this slope through the origin, so it is nilpotent. (In the proof of the periodicity theorem, the nilpotence theorem was needed at this point. We do not need an analog of it here because we have the vanishing line.) The methods of [Rav92a, 6.1.2] can be applied here to show that for some $i > 0$, F^{p^i} and G^{p^i} commute. Now replace F and G by their commuting powers. $F - G$ is still nilpotent for the same reason as before, and for $j \gg 0$ we have

$$0 = (F - G)^{p^j} = F^{p^j} - G^{p^j}.$$

Thus $F^{p^j} = G^{p^j}$. Replacing the original f and g by suitable powers we get a commutative diagram (ignoring suspensions)

$$\begin{array}{ccc} X \wedge y(n) & \xrightarrow{f \wedge y(n)} & X \wedge y(n) \\ {\scriptstyle X \wedge g} \downarrow & & \downarrow {\scriptstyle X \wedge g} \\ X \wedge y(n) & \xrightarrow{f \wedge y(n)} & X \wedge y(n). \end{array}$$

It follows that

$$\widehat{X} \wedge y(n) = X \wedge Y(n) = \widehat{X} \wedge Y(n).$$

Thus the map $y(n) \to Y(n)$ is an $\widehat{X}_*$-equivalence. The result will follow if we can show that $Y(n)$ is $\widehat{X}_*$-local. We have $Y(n) \wedge C_f = 0$, C_f being the cofiber of f and therefore a finite complex of type $n+1$. Given this, is follows from the thick subcategory theorem that $Y(n)$ annihilates all finite $K(n)_*$-acyclic complexes, so it is $\widehat{X}_*$-local. □

Conjecture 3.9. *$Y(n)$ has the same Bousfield class as the telescope associated with a finite complex of type n.*

This could be regarded as a new formulation of the telescope conjecture, with $Y(n)$ taking the place of $K(n)$. See also Conjecture 4.4 below. A stronger conjecture is the following.

Conjecture 3.10. *$Y(n)$ has the same homotopy type as an infinite wedge of finite type n telescopes.*

3.2. **The homotopy of $L_n y(n)$ and $Y(n)$.** We have

$$BP_*(L_n y(n)) = v_n^{-1} BP_*(y(n)) = v_n^{-1} BP_*/I_n[t_1, \ldots, t_n].$$

and we know that the Adams-Novikov spectral sequence converges to $\pi_*(L_n y(n))$. Its E_2-term was given above in 2.11, namely

$$E_2 = K(n)_*[v_{n+1}, \ldots, v_{2n}] \otimes E(h_{n+i,j} \colon 1 \leq i \leq n, 0 \leq j \leq n-1).$$

It follows from 3.5 that each v_{n+i} is a permanent cycle, as is $h_{n+i,j}$ for $i + j \leq n$. This accounts for just over half of the n^2 exterior generators. Perhaps other exterior generators are permanent cycles for dimensional reasons, but we will see below that similar elements in the localized Adams spectral sequence are not.

Question 3.11. *Does the Adams-Novikov spectral sequence for $L_n y(n)$ collapse?*

It does for sparseness reasons when $2p > n^2$. In any case we have the following result.

Corollary 3.12. *$\pi_*(L_n y(n))$ is finitely presented as a module over the ring*

$$R(n)_* = K(n)_*[v_{n+1}, \ldots, v_{2n}]. \tag{3.13}$$

We had hoped to show this is *not* true of $\pi_*(Y(n))$ for $n > 1$, showing that $Y(n)$ (which is $L_n^f y(n)$) differs from $L_n y(n)$, thereby disproving the telescope conjecture.

We can compute $\pi_*(Y(n))$ with the localized Adams spectral sequence. Its E_2-term was identified in (2.20) as

$$\begin{aligned} E_2 &= R(n)_* \otimes E(h_{n+i,j} : i > 0, 0 \leq j \leq n-1) \\ &= \qquad \otimes P(b_{n+i,j} : i > 0, 0 \leq j \leq n-1). \end{aligned}$$

As remarked above, the $h_{n+i,j}$ for $i+j \leq n$ and the v_{n+i} are permanent cycles.

Conjecture 3.14. *For $i > 0$ and $0 \leq j \leq n-1$, the element $h_{2n+i-j,j}$ survives to E_{2p^j} and supports a nontrivial differential*

$$d_{2p^j}(h_{2n+i-j,j}) = v_n b_{n+i,n-1-j}^{p^j}.$$

Each $b_{n+i,j}$ for $i > 0$ and $0 \leq j \leq n-2$ survives to $E_{1+2p^{n-1}}$.

Note that if in addition each $b_{n+i,j}$ were a permanent cycle, then we would have

$$E_\infty = R(n)_* \otimes E(h_{n+i,j} : i + j \leq n) \otimes P(b_{n+i,j})/(b_{n+i,j}^{p^{n-1-j}}). \tag{3.15}$$

For $n > 1$, this E_∞ and hence $\pi_*(Y(n))$ would be infinitely generated as a module over $R(n)_*$, which is incompatible with the telescope conjecture. However we cannot prove that each $b_{n+i,j}$ is a permanent cycle for $n > 1$, and it seems unlikely to be true. Hence we expect E_∞ to be more complicated than indicated by (3.15).

We have

Conjecture 3.16 (Differentials conjecture)**.** *In the localized Adams spectral sequence for $Y(n)$ the elements $h_{n+i,0}$ and $h_{n+i,1}$ survive to E_2 and E_{2p} respectively, and there are differentials*

$$\begin{aligned} d_2(h_{2n+i,0}+s_{2n+i,0}) &= v_n b_{n+i,n-1} \\ \textit{and} \qquad d_{2p}(h_{2n+i-1,1}+s_{2n+i-1,1}) &= v_n b_{n+i,n-2}^p \end{aligned}$$

for decomposables $s_{2n+i-j,j}$. The elements $b_{n+i,j}$ for $j<n-1$ survive to E_{2p+1}, so

$$\begin{aligned} E_{2p+1} = \; & R(n)_* \otimes E(h_{n+i,0} : 1 \le i \le n) \otimes E(h_{n+i,1} : 0 \le i \le n-1) \\ & \otimes E(h_{n+i,j} : i > 0, 2 \le j \le n-1) \\ & \otimes P(b_{n+i,n-2} : i > 0)/(b_{n+i,n-2}^p) \\ & \otimes P(b_{n+i,j} : i > 0, 0 \le j \le n-3). \end{aligned}$$

This will be discussed in §5. The strategy is to lift the computation back to a localized Thomified Eilenberg-Moore spectral sequence converging to $Y(n)_*(\Omega^3 S^{1+2p^n})$ in a manner to be described in §3.3, specifically using the map of (3.18) below. Curiously, its E_2-term is essentially the one above tensored with itself. The corresponding statement about differentials there is Conjecture 5.15.

The advantage of this lifting is that the localized Thomified Eilenberg-Moore spectral sequence has far more structure than the localized Adams spectral sequence above, due in large part to the structure of the space $\Omega^3 S^{1+2p^n}$. Its properties are developed in §4. It is an H-space (which makes the spectral sequence one of Hopf algebras) with a Snaith splitting described in §4.1. The pth Hopf map induces an endomorphism of the spectral sequence that is described in Lemma 5.16.

For $n=1$ Conjecture 3.16 gives the following.

$$E_2 = R(1)_* \otimes E(h_{2,0}, h_{3,0}, \ldots) \otimes P(b_{2,0}, b_{3,0}, \ldots)$$

with differentials

$$d_2(h_{i+1,0}) = v_1 b_{i,0} \qquad \text{for } i \ge 2,$$

which leaves

$$E_3 = E_\infty = R(1)_* \otimes E(h_{2,0}).$$

Thus for $n=1$, the localized Adams spectral sequence and the Adams-Novikov spectral sequence give the same answer. Miller [Mil81] proved the telescope conjecture for $n=1$ and p odd by doing a similar calculation with $y(1)$ replaced by $V(0)$.

For $n=2$ we have

$$E_2 = R(2)_* \otimes E(h_{3,0}, h_{3,1}, h_{4,0}, h_{4,1}, \ldots) \otimes P(b_{3,0}, b_{3,1}, b_{4,0}, b_{4,1}, \ldots).$$

The first differential,

$$d_2(h_{i+2,0}) = v_2 b_{i,1} \qquad \text{for } i \geq 3$$

gives

$$E_3 = R(2)_* \otimes E(h_{3,0}, h_{4,0}) \otimes E(h_{3,1}, h_{4,1}, \ldots) \otimes P(b_{3,0}, b_{4,0}, \ldots).$$

A pattern of higher differentials consistent with the telescope conjecture would be

$$d_{1+p^{i+1}}(h_{i+2,1}) = v_2^{p^{i+1}} b_{i,0} \qquad \text{for } i \geq 3.$$

Notice that these get arbitrarily long for large i, and they are preempted by the differentials of 3.16,

$$d_{2p}(h_{i+1,1}) = v_2 b_{i,0}^p.$$

The splitting of Lemma 3.4 has implications for the spectral sequences we are studying. For any space or spectrum X, $y(n)_*(X)$ is a left comodule over

$$y(n)_*(y(n)) = H_*(y(n)) \otimes y(n)_*.$$

The same goes for E_r of a spectral sequence converging to $y(n)_*(X)$, in which case $y(n)_*(X)$ may be filtered in a way compatible with this comodule structure. Similarly $Y(n)_*(X)$ is a left comodule over

$$Y(n)_*(Y(n)) = H_*(y(n)) \otimes Y(n)_*,$$

so that any spectral sequence converging to $Y(n)_*$ has a comodule structure over $Y(n)_*Y(n)$.

Lemma 3.17. *The comodule structure of the localized E_2-term of (2.20) is given by*

$$\begin{aligned} \psi(v_{n+i}) &= \sum_{0 \leq k \leq i} \xi_{i-k}^{p^{n+k}} \otimes v_{n+k}, \\ \psi(h_{n+i,j}) &= \sum_{0 \leq k \leq n-1-j} \overline{\xi}_k^{p^j} \otimes h_{n+i-k,j+k} \\ \textit{and} \qquad \psi(b_{n+i,j}) &= \sum_{0 \leq k \leq n-1-j} \overline{\xi}_k^{p^{j+1}} \otimes b_{n+i-k,j+k}. \end{aligned}$$

Further, in the localized Adams spectral sequence for $Y(n)$, if x is any of the $h_{n+i,j}$ or $b_{n+i,j}$ and $d_2(x)$ is nontrivial, it cannot be divisible by v_{n+k} for any $k > 0$.

Proof. The coalgebra structure in $H_*(y(n))$ can be read off by injecting it into the dual Steenrod algebra. The values of $\psi(v_{n+i})$ and $\psi(h_{n+i,j})$ can be read off from the coproducts on τ_{n+i} and $\overline{\xi}_{n+i}^{p^j}$ in A_*, and $\psi(b_{n+i,j})$ is the transpotent of the latter.

The divisibility of $d_2(x)$ by v_{n+k} would contradict this comodule structure. □

3.3. **The triple loop space.** Now we will explain why the triple loop space $\Omega^3 S^{1+2p^n}$ is relevant to the proof of the differentials conjecture. Consider the following diagram in which each row is a fiber sequence.
(3.18)

$$\begin{array}{ccccc} \Omega J_{p^n-1}S^2 \times \Omega^3 S^{1+2p^n} & \longrightarrow & \Omega^2 S^3 & \longrightarrow & \Omega^2 S^{1+2p^n} \times \Omega^2 S^{1+2p^n} \\ \downarrow & & \| & & \downarrow \\ \Omega J_{p^n-1}S^2 & \xrightarrow{i} & \Omega^2 S^3 & \longrightarrow & \Omega^2 S^{1+2p^n} \end{array}$$

Here the top row is the Cartesian product of the bottom row with the path fibration on $\Omega^2 S^{1+2p^n}$. The right vertical map is loop space multiplication, while the left one is the product of the identity map on the first factor with the inclusion of the fiber of i on the second factor.

We will look at the Thomified Eilenberg-Moore spectral sequence for each row where the spherical fibration over $\Omega^2 S^3$ is the one given by 3.3. Then the bottom row satisfies the hypotheses of Theorem 2.26(ii), so we get the Adams spectral sequence for $y(n)$.

For the top row, the E_2-term is described by the following specialization of 2.26.

Theorem 3.19. *Consider the Thomified Eilenberg-Moore spectral sequence associated with $E = \Omega^2 S^3$, equipped with the spherical fibration given by 3.3. Suppose the defining fibration has the form*

$$\begin{array}{ccccc} X & \xrightarrow{i} & E & \xrightarrow{h} & B \\ \| & & \| & & \| \\ X_1 \times \Omega B_2 & \xrightarrow{i_1 \times i_2} & \Omega^2 S^3 \times \mathrm{pt.} & \xrightarrow{h_1 \times *} & B_1 \times B_2 \end{array}$$

where h is an H-map and $H_*(i_1)$ is monomorphic, and $Y = Y_1 \wedge \Omega B_{2+}$. Then $H_*(Y_1)$ is a subalgebra of $A_* = H_*(K)$, and we let

$$\Gamma = A_* \otimes_{H_*(Y_1)} \mathbf{Z}/(p)$$

Then the E_2-term of the Thomified Eilenberg-Moore spectral sequence is

$$\mathrm{Ext}_{H_*(B_2) \otimes \Gamma}(\mathbf{Z}/(p), \mathbf{Z}/(p)),$$

where $H_*(B_2) \otimes \Gamma$ *is a semitensor product coalgebra with coproduct as in (2.23).*

In the next section we will see that the top row of (3.18) satisfies the hypotheses of 3.19. In this case the Hopf algebra Γ is $B(n)_*$ of (2.2).

Proof. We have a Hopf algebra extension

$$H_*(B_2) \otimes A_* \longrightarrow H_*(B_1) \otimes H_*(B_2) \otimes A_* \longrightarrow H_*(B_1)$$

and hence a Cartan-Eilenberg spectral sequence converging to the E_2 of the Thomified Eilenberg-Moore spectral sequence with

$$E_2 = \mathrm{Ext}_{H_*(B_2)\otimes A_*}(\mathbf{Z}/(p), \mathrm{Ext}_{H_*(B_1)}(\mathbf{Z}/(p), H_*(K))).$$

In our case $H_*(K)$ is a free comodule over $H_*(B_1)$, so the prespectral sequence collapses to

$$\mathrm{Ext}_{H_*(B_2)\otimes A_*}(\mathbf{Z}/(p), H_*(Y_1)). \tag{3.20}$$

Using the Hopf algebra extension

$$H_*(B_2) \otimes \Gamma \longrightarrow H_*(B_2) \otimes A_* \longrightarrow H_*(Y_1)$$

we can equate (3.20) with

$$\mathrm{Ext}_{H_*(B_2)\otimes\Gamma}(\mathbf{Z}/(p), \mathbf{Z}/(p)).$$

as claimed. □

Theorem 3.21. *The Thomified Eilenberg-Moore spectral sequence for the top row of (3.18) can be localized in the same way that the one for the bottom row can.*

Proof. The spectral sequence in question is based on the diagram (2.22) with $Y_0 = y(n) \wedge \Omega^3 S_+^{1+2p^n}$. This diagram has suitable multiplicative properties. In order to get a localized resolution as in (2.12), we need to lift the map $v_n \wedge \Omega^3 S_+^{1+2p^n}$ to Y_1. This lifting exists if and only if $g_0(v_n \wedge \Omega^3 S_+^{1+2p^n})$ is null, which it is since $K_0 = H/p$ and $H_*(v_n) = 0$.

Thus the Thomified Eilenberg-Moore spectral sequence for the top row of (3.18) can be localized compatibly with our localization of the Adams spectral sequence associated with the bottom row. Convergence of the localization of the top row (which is not actually needed for our purposes) can be proved using the argument of Theorem 2.13 provided we have a suitable vanishing line. Our E_2-term is a subquotient of

$$\mathrm{Ext}_{B(n)_*}(\mathbf{Z}/(p), \mathrm{Ext}_{H_*(\Omega^2 S^{1+2p^n})}(\mathbf{Z}/(p), \mathbf{Z}/(p))).$$

The connectivities of $B(n)_*$ and $\Omega^2 S^{1+2p^n}$ imply that both factors have a vanishing line of slope $1/|v_n|$ as required. □

4. Properties of $\Omega^3 S^{1+2p^n}$

4.1. **The Snaith splitting.** For each $n > 0$ we have a fibration of spaces (3.1)

$$J_{p^n-1}S^2 \to \Omega S^3 \to \Omega S^{1+2p^n}$$

which leads to a stable map

$$\Omega^3 S_+^{1+2p^n} \xrightarrow{f} y(n). \tag{4.1}$$

We know that $\Omega^3 S_+^{1+2p^n}$ has a Snaith splitting [Sna74]

$$\Omega^3 S_+^{1+2p^n} \simeq \bigvee_{i \geq 0} \Sigma^{|v_n|i} T_i. \tag{4.2}$$

Here T_i is a certain finite complex (independent of n) with bottom cell in dimension 0 and top cell in dimension $2i - 2\alpha(i)$, where $\alpha(i)$ denotes the sum of the digits in the p-adic expansion of i. In particular $T_1 = S^0$.

Moreover there are pairings

$$T_i \wedge T_j \to T_{i+j}.$$

Thus we get a ring spectrum

$$T_\infty = \lim_{\to} T_i. \tag{4.3}$$

Using the map v_n of (3.7) and the map f of (4.1), for each $i \geq 0$ we get a diagram

$$\begin{array}{ccccccc}
\Sigma^{i|v_n|}T_i & \longrightarrow & \Sigma^{i|v_n|}T_{i+1} & \longrightarrow & \cdots & \longrightarrow & \Sigma^{i|v_n|}T_\infty \\
\Big\downarrow f & & \Big\downarrow f & & & & \Big\downarrow \widehat{f} \\
y(n) & \xrightarrow{v_n} & \Sigma^{-|v_n|}y(n) & \xrightarrow{v_n} & \cdots & \longrightarrow & Y(n)
\end{array}$$

The map f on the left is v_n^i on the bottom cell of its source. The horizontal maps in the top row each multiply the bottom cell by v_n, so that each square in the diagram commutes. The map $\widehat{f}$ on the right makes $Y(n)$ a module spectrum over T_∞. If 3.9 is true, then the following seems likely.

Conjecture 4.4. *The Bousfield class of T_∞ is that of the p-local sphere spectrum.*

T_∞ is also the Thom spectrum of a bundle over $\Omega_0^3 S^3$ obtained as follows. From the EHP sequence we get a fiber sequence

$$\Omega^2 S^{2p-1} \longrightarrow \Omega_0^3 S^3 \longrightarrow \Omega^3 S^{2p+1}.$$

We get our bundle from one over $\Omega^3 S^{2p+1}$ obtained by extending the map $S^{2p-2} \to BU$ corresponding to a generator of $\pi_{2p-2}(BU)$.

Equivalently, our bundle is the one obtained from the map

$$\Omega_0^3 S^3 = \Omega_0^3 SU(2) \longrightarrow \Omega_0^3 SU = BU.$$

There is also a Hopf map

$$\Omega^3 S^{1+2p^n} \xrightarrow{H} \Omega^3 S^{1+2p^{n+1}} \tag{4.5}$$

which is surjective in ordinary homology. It induces a map from the (pi)th Snaith summand of the source to the ith one of the target,

$$T_{pi} \xrightarrow{H} \Sigma^{2(p-1)i} T_i, \tag{4.6}$$

which has degree one on the top cell (in dimension $2pi - 2\alpha(i)$). We will use this map to study $\Omega^3 S^{1+2p^n}$ and T_∞ below.

Recall (4.3) that the spectrum T_∞ is the homotopy direct limit of the Snaith summands of a certain triple loop space. The analogous spectrum obtained from the Snaith splitting of the double loop space of an odd dimensional sphere is H/p, but T_∞ is far more interesting. It turns out that $K(n)_*(T_\infty)$ bears a remarkable resemblance to the supposed value of $\pi_*(Y(n))$. (Compare Conjecture 3.16 and Theorem 4.17 below.)

4.2. **Ordinary homology.** $H_*(\Omega^3 S^{2dp+1})$ has long been known [CLM76] as a module over the Steenrod algebra A, and is as follows.

Lemma 4.7.

$$H_*(\Omega^3 S^{2dp+1}) = \begin{cases} P(u_i : i \geq 0) \otimes P(x_{i,j} : i > 0, j \geq 0) & \text{for } p = 2 \\ P(u_i : i \geq 0) \otimes E(x_{i,j} : i > 0, j \geq 0) & \\ \qquad\qquad \otimes P(y_{i,j} : i > 0, j \geq 0) & \text{for } p > 2 \end{cases}$$

where $|u_i| = 2(p^{i+1}d - 1)$, $|x_{i,j}| = 2p^j(p^{i+1}d - 1) - 1$ *and* $|y_{i,j}| = 2p^{j+1}(p^{i+1}d - 1) - 2$.

For all primes this group can be identified with

$$\mathrm{Ext}_{H_*(\Omega^2 S^{1+2dp})}(\mathbf{Z}/(p), \mathbf{Z}/(p)),$$

i.e., the Eilenberg-Moore spectral sequence in mod p homology for the path fibration on $\Omega^2 S^{1+2dp}$ *collapses.*

For $p = 2$ the action of the Steenrod algebra A is given by

$$\begin{aligned}
\mathrm{Sq}_*^{2^k}(u_i) &= \begin{cases} x_{i,0} & \textit{if } k = 0 \\ u_{i-1}^2 & \textit{if } k = 1 \\ x_{i-k+1,k-2}^2 & \textit{otherwise} \end{cases} \\
\mathrm{Sq}_*^{2^k}(x_{i,j}) &= \begin{cases} x_{i,j-1}^2 & \textit{if } k = 0 \textit{ and } j > 0 \\ x_{i-1,j+1} & \textit{if } k = j+1 \\ 0 & \textit{otherwise.} \end{cases}
\end{aligned}$$

For p odd we have

$$\begin{aligned}
\beta_*(u_i) &= x_{i,0} \\
P_*^{p^k}(u_i) &= \begin{cases} -u_{i-1}^p & \textit{if } k = 0 \\ 0 & \textit{otherwise} \end{cases} \\
\beta_*(x_{i,j}) &= y_{i,j-1} \qquad \textit{for } j > 0 \\
P_*^{p^k}(x_{i,j}) &= \begin{cases} x_{i-1,j+1} & \textit{if } k = j \\ 0 & \textit{otherwise} \end{cases} \\
\beta_*(y_{i,j}) &= 0 \\
P_*^{p^k}(y_{i,j}) &= \begin{cases} -y_{i,j-1}^p & \textit{if } k = 0 \textit{ and } j > 0 \\ y_{i-1,j+1} & \textit{if } k = j+1 \\ 0 & \textit{otherwise.} \end{cases}
\end{aligned}$$

We will also need to know the action of the Milnor primitives Q_k, which can be read off from Lemma 4.7. Up to sign we have

$$\begin{aligned}
Q_k(u_i) &= \begin{cases} x_{i-k,k} & \text{if } k < i \\ 0 & \text{otherwise} \end{cases} \\
Q_k(x_{i,j}) &= \begin{cases} y_{i,j-k-1}^{p^k} & \text{for } 0 \le k < j \\ 0 & \text{for } k = j \\ y_{i+j-k,k-j-1}^{p^j} & \text{for } j < k < i+j \end{cases} \qquad (4.8)
\end{aligned}$$

where $y_{i,j} = x_{i,j}^2$ when $p = 2$.

Proof of Lemma 4.7. We will prove this for p odd, leaving the case $p = 2$ (which is easier) as an exercise for the reader. We will relate the description of the Lemma to the one given by Cohen in [CLM76]. There he speaks of Dyer-Lashof operations with upper indices $Q^s : H_q \to H_{q+2(p-1)s}$. *Within this proof Q_s will denote a reindexed Dyer-Lashof operation rather than the Milnor operation.* We define $Q_s : H_q \to H_{pq+(p-1)s}$, when q and s have the same parity, by

$$Q_s = Q^{(s+q)/2}$$

with

$$Q_s(x) = \begin{cases} 0 & \text{for } s < 0 \\ x^p & \text{for } s = 0. \end{cases}$$

With this in mind, Cohen's result says that

$$\begin{aligned} H_*(\Omega^3 S^{2pd+1}) &= P(Q_2^i(u_0) : i \geq 0) \otimes E(Q_1^j \beta Q_2^i(u_0) : i > 0, j \geq 0) \\ &\quad \otimes P(\beta Q_1^{j+1} \beta Q_2^i(u_0) : i > 0, j \geq 0), \end{aligned}$$

where $u_0 \in H_{2pd-2}$ is the fundamental class. We define

$$\begin{aligned} u_i &= Q_2^i(u_0), \\ x_{i,j} &= Q_1^j \beta Q_2^i(u_0) \\ \text{and} \qquad y_{i,j} &= \beta Q_1^{j+1} \beta Q_2^i(u_0). \end{aligned}$$

These elements have the indicated dimensions. It remains to show that the action of the Steenrod algebra is as stated.

The action of Steenrod operations on Dyer-Lashof operations is given by the Nishida relations. For operations on a q-dimensional class, these are

(4.9) $$P_*^r Q_s = \sum_i (-1)^{r+i} \binom{(p-1)\left(\frac{s+q}{2} - r\right)}{r - pi} Q_{s-2r+2pi} P_*^i$$

and

(4.10)
$$\begin{aligned} P_*^r \beta Q_s &= \sum_i (-1)^{r+i} \binom{(p-1)\left(\frac{s+q}{2} - r\right) - 1}{r - pi} \beta Q_{s-2r+2pi} P_*^i \\ &\quad + \sum_i (-1)^{r+i} \binom{(p-1)\left(\frac{s+q}{2} - r\right) - 1}{r - pi - 1} Q_{s+1-2r+2pi} P_*^i \beta. \end{aligned}$$

In particular we have

$$\begin{aligned} P_*^1 Q_2 &= \frac{q}{2} Q_0, \\ \text{so} \qquad P_*^1(u_i) &= P_*^1 Q_2(u_{i-1}) \\ &= -Q_0(u_{i-1}) \\ &= -u_{i-1}^p. \end{aligned}$$

For $k > 0$ (4.9) gives

$$\begin{aligned} P_*^{p^k} Q_2 &= Q_2 P_*^{p^{k-1}}, \\ \text{so} \qquad P_*^{p^k}(u_i) &= P_*^{p^k} Q_2^k(u_{i-k}) \\ &= Q_2^k P_*^1(u_{i-k}) \\ &= -Q_2^k(u_{i-k-1}^p) \\ &= 0 \qquad \text{by the Cartan formula.} \end{aligned}$$

We have $\beta(u_i) = x_{i,0}$ and $\beta(x_{i,j+1}) = y_{i,j}$ by definition, and it follows that $\beta(x_{i,0}) = 0$ and $\beta(y_{i,j}) = 0$.

The Nishida relations also give

$$P_*^{p^k} Q_1 = \begin{cases} 0 & k = 0 \\ Q_1 P_*^{p^{k-1}} & k > 0, \end{cases}$$

and for $s = 1$ or 2

$$P_*^{p^k} \beta Q_s = \begin{cases} -Q_{s-1}\beta & k = 0 \\ \beta Q_s P_*^{p^{k-1}} & k > 0. \end{cases}$$

It follows that

$$\begin{aligned} P_*^1(x_{i,0}) &= P_*^1 \beta Q_2^i(u_0) \\ &= -Q_1 \beta Q_2^{i-1}(u_0) \\ &= -x_{i-1,1}, \end{aligned}$$

and for $k > 0$

$$\begin{aligned} P_*^{p^k}(x_{i,0}) &= P_*^{p^k} \beta Q_2^i(u_0) \\ &= \begin{cases} \beta Q_2^k P_*^1 Q_2^{i-k}(u_0) & k < i \\ \beta Q_2^i P_*^{p^{k-i}}(u_0) & k \geq i \end{cases} \\ &= 0. \end{aligned}$$

For $j > 0$,

$$\begin{aligned} P_*^1(x_{i,j}) &= P_*^1 Q_1^j \beta Q_2^i(u_0) \\ &= 0, \end{aligned}$$

and when $k > 0$

$$\begin{aligned} P_*^{p^k}(x_{i,j}) &= P_*^{p^k} Q_1^j(x_{i,0}) \\ &= \begin{cases} 0 & k < j \\ Q_1^j P_*^{p^{k-j}}(x_{i,0}) & k \geq j \end{cases} \\ &= \begin{cases} 0 & k < j \\ Q_1^j(x_{i-1,1}) & k = j \\ 0 & k > j \end{cases} \\ &= \begin{cases} x_{i-1,j+1} & k = j \\ 0 & k \neq j \end{cases} \end{aligned}$$

as claimed.

Finally we have for $j > 0$,

$$\begin{aligned} P_*^1(y_{i,j}) &= P_*^1 \beta Q_1^{j+1}(x_{i,0}) \\ &= -Q_0 \beta Q_1^j(x_{i,0}) \\ &= -Q_0(y_{i,j-1}) \\ &= -y_{i,j-1}^p, \end{aligned}$$

and for all j when $k > 0$

$$\begin{aligned} P_*^{p^k}(y_{i,j}) &= P_*^{p^k} \beta Q_1^{j+1}(x_{i,0}) \\ &= \beta Q_1 P_*^{p^{k-1}} Q_1^j(x_{i,0}) \\ &= \beta Q_1 P_*^{p^{k-1}}(x_{i,j}) \\ &= \begin{cases} 0 & k-1 \neq j \\ \beta Q_1(x_{i-1,j+1}) & k-1 = j \end{cases} \\ &= \begin{cases} 0 & k \neq j+1 \\ y_{i-1,j+1} & k = j+1 \end{cases} \end{aligned}$$

as claimed. □

4.3. **Morava K-theory.** In this subsection we will study the Eilenberg-Moore spectral sequence for $K(n)_*(\Omega^3 S^{2dp+1})$ for $d > 0$. First we need to know $K(n)_*(\Omega^2 S^{2dp+1})$, which was computed by Yamaguchi [Yam88]. We will assume for simplicity that p is odd. We could find $K(n)_*(\Omega^2 S^{2dp+1})$ with either the Atiyah-Hirzebruch spectral sequence or the Eilenberg-Moore spectral sequence starting with $K(n)_*(\Omega S^{2pd+1})$. (Since ΩS^{2pd+1} splits after a single suspension, it is easy to work out its Morava K-theory.) It turns out that the two spectral sequences are the same up to reindexing, and we will describe the former.

We have

$$H_*(\Omega^2 S^{2dp+1}) = E(e_i : i \geq 0) \otimes P(f_i : i \geq 0) \tag{4.11}$$

with $|e_i| = 2dp^i - 1$ and $|f_i| = 2dp^{i+1} - 2$. (When $p = 2$, $f_i = e_i^2$.) In terms of the Dyer-Lashof operations Q_i we have

$$\begin{aligned} e_i &= Q_1^i(e_0) \\ \text{and} \qquad f_i &= \beta Q_1^{i+1}(e_0) \end{aligned}$$

The coaction of the dual Steenrod algebra A_* is given by

$$\begin{aligned} e_i &\mapsto 1 \otimes e_i + \textstyle\sum_{0 \le k < i} \overline{\tau}_{n+k} \otimes f_{i-n-k}^{p^{n+k}} \\ \text{and} \qquad f_i &\mapsto \textstyle\sum_{0 \le k < i} \overline{\xi}_k \otimes f_{i-k}^{p^k}. \end{aligned} \tag{4.12}$$

In the Atiyah-Hirzebruch spectral sequence there are differentials

$$d_{1+|v_n|}(e_{i+n+1}) = v_n f_i^{p^n} \tag{4.13}$$

determined by the Milnor operation Q_n. This leaves

$$E_{2p^n} = K(n)_* \otimes E(e_0, e_1, \cdots e_n) \otimes P(f_i)/(f_i^{p^n}).$$

Yamaguchi has shown that there are no higher differentials or multiplicative extensions. It is useful to reprove his theorem here.

Theorem 4.14. *With notation as above,*

$$K(n)_*(\Omega^2 S^{2dp+1}) = K(n)_* \otimes E(e_0, e_1, \cdots e_n) \otimes P(f_i)/(f_i^{p^n}).$$

Moreover, the elements e_i for $0 \le i \le n$ and f_i for all $i \ge 0$ are in the image of $k(n)_*(\Omega^2 S^{2dp+1})$.

Proof. We need to show that there are no higher differentials or multiplicative extensions. We know that $\Omega^2 S^{2dp+1}$ has a stable splitting which must be respected by all differentials. The Snaith degrees of e_i and f_{i-1} are each p^i.

We also know that we have a spectral sequence of Hopf algebras, using the Kunneth isomorphism for $K(n)_*$. Hence, differentials must send primitives to primitives. E_{2p^n} is primitively generated. The set of primitives with Snaith degree p^m is a subset (depending on m) of

$$\{e_m, f_{m-1}, f_{m-2}^p, \cdots, f_{m-n}^{p^{n-1}}\}.$$

The dimensions of these elements are all within $|v_n|$ of each other, so there is no room for higher differentials, and $E_\infty = E_{2p^n}$.

To show there are no multiplicative extensions, we must show that $f_i^{p^n} = 0$ in $K(n)_*(\Omega^2 S^{2dp+1})$. If this element were nonzero it would be a primitive with Snaith degree p^{i+n+1}. This means it would have to be a linear combination (with coefficients in $k(n)_*$) of the elements

$$\{f_{i+n}, f_{i+n-1}^p, \cdots, f_{i+1}^{p^{n-1}}\}.$$

These elements all have dimensions higher than that of $f_i^{p^n}$, so the latter must be zero as claimed.

For the statement about $k(n)_*(\Omega^2 S^{2dp+1})$, note that in the Atiyah-Hirzebruch spectral sequence for this group, the indicated elements cannot support any nontrivial differentials for the same dimensional reason as before. □

In order to use the Eilenberg-Moore spectral sequence (for which the E_2-term is Cotor), we need to know the coalgebra structure. This can be described by giving the Verschiebung map V, which is dual to the pth power map. In [Rav93a], the second author proved that

$$V(v_n^{-1} f_{i+n}) = f_i^{p^{n-1}}. \tag{4.15}$$

For the purposes of this calculation, it is convenient to ignore this Verschiebung and proceed as if the Hopf algebra were primitively generated. The resulting Cotor group can be regarded as the E_1-term of Tamaki's spectral sequence, and there will be some d_1s reflecting the behavior of V. In other words we can use the Eilenberg-Moore filtration of $K(n)_*(\Omega^2 S^{2pd+1})$ to set up a spectral sequence converging to the desired Cotor group.

With this in mind, let u_i denote the desuspension of e_i for $0 \leq i \leq n$, let $x_{i,j}$ denote that of $f_i^{p^j}$ for $i \geq 0$ and $0 \leq j < n$, and let $y_{i,j}$ be the transpotent of same. In terms of Dyer-Lashof operations on the corresponding classes in ordinary homology, we have

$$\begin{array}{rrclcl} & u_i & = & \tau(Q_1^i(e_0)) & = & Q_2^i(u_0), \\ & x_{i,j} & = & \tau(Q_0^j \beta Q_1^i(e_0)) & = & Q_1^j \beta Q_2^i(u_0) \\ \text{and} & y_{i,j} & = & \kappa(Q_0^j \beta Q_1^i(e_0)) & = & \beta Q_1^{j+1} \beta Q_2^i(u_0) \end{array} \tag{4.16}$$

where τ denotes transgression (or desuspension) and κ denotes Kudo transgression, or transpotence.

Under our indexing conventions the Eilenberg-Moore spectral sequence is in the first quadrant with $u_i, x_{i,j} \in E_1^{1,*}$ and $y_{i,j} \in E_1^{2,*}$, and d_r raises the first index by r. We have

$$E_1 = P(u_k : 0 \leq k \leq n) \otimes E(x_{i,j} : j < n) \otimes P(y_{i,j} : j < n).$$

(In the corresponding ordinary mod p homology Eilenberg-Moore spectral sequence, the E_2-term has a similar description, but without upper bounds on the subscripts, and there are no differentials.) It is a spectral sequence of Hopf algebras (again, because of the Kunneth isomorphism for $K(n)_*$,) so differentials must send primitives to primitives. Note that the $x_{i,j}$s above are odd dimensional while the other generators and $K(n)_*$ are even dimensional.

From (4.15) we get

$$d_1(x_{i+n,0}) = v_n y_{i,n-1}$$

(compare this with (4.8) for $j = 0$ and $k = n$) so we have

$$\begin{aligned} E_2 &= P(u_i : 0 \le i \le n) \otimes E(x_{i,0} : i \le n) \\ &\quad \otimes P(y_{i,j} : j \le n-2) \otimes E(x_{i,j} : 1 \le j \le n-1). \end{aligned}$$

As in the case of the Atiyah-Hirzebruch spectral sequence above, we can take advantage of the Snaith splitting and the Hopf algebra structure. We can also take advantage of the Hopf map of (4.5) and (4.6).

Theorem 4.17. *In the Eilenberg-Moore spectral sequence for $K(n)_*(\Omega^3 S^{2dp+1})$, we have*

$$d_{2p^j-1}(x_{i+n,j}) = v_n y_{i+j,n-1-j}^{p^j} \qquad \textit{for } 1 \le j \le n-1.$$

No other differentials occur and $E_\infty = E_{2p^{n-1}}$.

For odd primes there are no multiplicative extensions,

$$\begin{aligned} K(n)_*(\Omega^3 S^{2dp+1}) &= K(n)_*[u_0, \cdots u_n] \otimes E(x_{i,j} : i+j \le n) \\ &\quad \otimes P(y_{i,j} : j \le n-2)/(y_{i,j}^{p^{n-j-1}}). \end{aligned}$$

Proof. The primitives in E_2 with Snaith degree p^m for $m > n$ are

$$(4.18) \qquad \begin{array}{c} \{x_{m-j,j} \colon 1 \le j \le n-1\} \quad \cup \\ \{y_{m-\ell-k-1,k}^{p^\ell} \colon 0 \le k \le n-2, 0 \le \ell + k \le m-2\} \quad \cup \\ \{u_k^{p^{m-k}} : 0 \le k \le n\}; \end{array}$$

for $1 \le m \le n$ we have to add the element $x_{m,0}$.

The dimensions of these elements are

$$(4.19) \qquad \begin{cases} |x_{m-j,j}| = 2dp^m - 2p^j - 1, \\ |y_{m-\ell-k-1,k}^{p^\ell}| = 2dp^m - 2p^{\ell+k+1} - 2p^\ell \\ \text{and} \quad |u_k^{p^{m-k}}| = 2dp^m - 2p^{m-k} \end{cases}$$

For $m \le n$, these dimensions are all within $|v_n|$ of each other, so there can be no higher differentials in these Snaith degrees. In particular the elements u_k for $0 \le k \le n$, $x_{i,j}$ for $i+j \le n$ and $y_{i,j}$ for $i+j < n$ are all permanent cycles.

For $m = n+1$, (4.18) reads

$$\{x_{n+1-j,j}\colon 1 \le j \le n-1\} \quad \cup$$
$$\{y_{n-j-k,j}^{p^k}\colon 0 \le j \le n-2, 0 \le j+k \le n-1\} \quad \cup$$
$$\{u_k^{p^{n+1-k}}\colon 0 \le k \le n\}.$$

The dimensions of these are compatible with the desired differentials

$$d_{2p^j-1}(x_{n+1-j,j}) = v_n y_{1,n-1-j}^{p^j} \qquad \text{for} 1 \le j \le n-1, \tag{4.20}$$

which can be inferred from (4.8). The only remaining primitives in this Snaith degree are even dimensional, all of the odd dimensional ones having been accounted for.

Now consider the primitives in Snaith degree p^{n+i} for $i > 1$. We claim that the only differentials that occur here are

$$d_{2p^j-1}(x_{i+n-j,j}) = v_n y_{i,n-j-1}^{p^j} \quad \text{for} \quad j \le n-1. \tag{4.21}$$

Here we make use of the Hopf map H. Its $(i-1)$th iterate is

$$\Omega^3 S^{1+2p^n} \xrightarrow{H^i} \Omega^3 S^{1+2p^{n+i+1}},$$

which induces a map

$$T_{pj} \xrightarrow{H^i} \Sigma^{2(p-1)ij} T_{ij}.$$

In the Eilenberg-Moore spectral sequence, the induced map sends the source and target of (4.21) to those of (4.20). Thus (4.21) holds *provided* that there is no earlier differential on $x_{i+n-j,j}$. Once these differentials have been taken into account, there are no odd dimensional primitives left in Snaith degrees above p^n, so the spectral sequence collapses from $E_{2p^{n-1}}$.

To prove (4.21) note that the elements of (4.19) with $\ell + k < n+1$ have dimensions too high to be a target of a differential on $x_{n+i-j,j}$, the ones with $\ell + k = n+1$ are the proposed targets, and the one with $\ell + k > n+1$ are powers of elements of lower Snaith degree that have already been killed. The elements $u_k^{p^{n+i-k}}$ have either too high a dimension (if $i > k$) or too high a filtration (if $i \le k$) to be the target of an earlier differential on $x_{n+i-j,j}$.

It follows that

$$\begin{aligned} E_\infty &= E_{2p^{n-1}} \\ &= K(n)_*[u_0, \cdots u_n] \otimes E(x_{i,j} : i+j \le n) \\ &\qquad \otimes P(y_{i,j} : j \le n-2)/(y_{i,j}^{p^{n-j-1}}) \end{aligned}$$

This spectral sequence converges to $K(n)_*(\Omega^3 S^{2dp+1})$.

To show there are no multiplicative extensions, we need to show that $y_{i,j}^{p^{n-1-j}} = 0$. If it is nonzero, it must be a $K(n)_*$-linear combination of the elements $u_k^{p^{n+i-k}}$ for $0 \le k \le n$, but the latter do not have the right dimensions modulo $|v_n|$, except possibly for $p = 2$. □

4.4. **The computation of $Y(n)_*(\Omega^3 S^{1+2p^n})$ via the Eilenberg-Moore spectral sequence.** In this subsection we will prove the following.

Theorem 4.22. *For each $n > 0$ there is an additive isomorphism*

$$Y(n)_*(\Omega^3 S^{2dp+1}) = Y(n)_* \otimes_{K(n)_*} K(n)_*(\Omega^3 S^{2dp+1}).$$

The isomorphism here need not be multiplicative. We will say more about this below after the proof. Before proving his result we need the following.

Lemma 4.23. *There is an additive isomorphism*

$$Y(n)_*(\Omega^2 S^{2dp+1}) = Y(n)_* \otimes_{K(n)_*} K(n)_*(\Omega^2 S^{2dp+1}).$$

Proof. Consider first the Atiyah-Hirzebruch spectral sequence for $y(n)_*(\Omega^2 S^{2dp+1})$. We have the differentials of (4.13), which leaves

$$E_{2p^n} = y(n)_* \otimes E(e_0, e_1, \cdots e_n) \otimes P(f_i)/(v_n f_i^{p^n}).$$

It follows that in the Atiyah-Hirzebruch spectral sequence for $Y(n)_*(\Omega^2 S^{2dp+1})$ we have

$$E_{2p^n} = Y(n)_* \otimes E(e_0, e_1, \cdots e_n) \otimes P(f_i)/(f_i^{p^n}).$$

Now we can argue as in the proof of 4.14 that there can be no higher differentials for dimensional reasons. □

Proof of Theorem 4.22. To prove the theorem we will use Tamaki's spectral sequence and the computation is essentially the same as that of §4.3. It is not necessary to know $\pi_*(Y(n))$ explicitly. In order to use Tamaki's spectral sequence we will use the computation of $Y(n)_*(\Omega^2 S^{2dp+1})$ of 4.23. The group $Y(n)_*(\Omega^2 S^{2dp+1})$ is a free $Y(n)_*$-module. This means that the E_2-term of the Tamaki spectral sequence can be identified as a Cotor group as in the Morava K-theory case.

In this setting, Tamaki's spectral sequence is one of Hopf algebras, despite the absence of a Kunneth isomorphism. The usual structure maps for X an iterated loop space on a sphere give homomophisms both ways between $Y(n)_*(X)$ and $Y(n)_*(X \times X)$. Further, we always have a Kunneth map $Y(n)_*(X) \otimes_{Y(n)_*} Y(n)_*(X) \to Y(n)_*(X)$. In this setting, however, the diagonal map $Y(n)_*(X) \to Y(n)_*(X \times X)$ need not factor through $Y(n)_*(X) \otimes_{Y(n)_*} Y(n)_*(X)$. We do have a Kunneth isomorphism for the E_2 term of the Tamaki spectral sequence, so that this E_2 term is a Hopf algebra, and the spectral sequence will be one of Hopf algebras, provided each E_r is free over $Y(n)_*$. The differentials are given below, and none of them introduce any $Y(n)_*$-torsion into any E_r, so we have the desired freeness.

In order to use the Tamaki spectral sequence we need to know the coalgebra structure. As in the computation for Morava K-theory, it is convenient to ignore the nontriviality of the Verschiebung and proceed as if the Hopf algebra were primitively generated. The resulting Cotor group can be regarded as the E_1-term of the Tamaki spectral sequence. It is generated as an algebra by elements in cohomological degrees 1 and 2, so the generators must be primitive. With this in mind, we can define u_i, $x_{i+1,j}$, and $y_{i+1,j}$ as in (4.16). As before we have

$$\begin{aligned} E_1 &= Y(n)_* \otimes P(u_i : 0 \le i \le n) \otimes E(x_{i,0} : i \le n) \\ &\quad \otimes P(y_{i,j} : j \le n-1) \otimes E(x_{i,j} : 0 \le j \le n-1). \end{aligned}$$

This Eilenberg-Moore spectral sequence maps to the one for Morava K-theory where we have the differentials of (4.21), namely

$$d_{2p^j-1}(x_{i+n-j,j}) = v_n y_{i,n-j-1}^{p^j} \quad \text{for} \quad j \le n-1.$$

Similar differentials will occur in the spectral sequence at hand if they are not preempted by earlier ones. In Snaith degrees $< p^{n+1}$ we can compare with the Eilenberg-Moore spectral sequence for $y(n)_*(\Omega^3 S^{2pd+1})$ and conclude that there are no differentials for dimensional reasons, so again the elements u_i for $i \le n$, $x_{i,j}$ for $i+j \le n$ and $y_{i,j}$ for $i+j < n$ are all permanent cycles.

In Snaith degree p^{n+1}, the primitives are

$$\begin{gathered} \{x_{n+1-j,j} : 0 \le j \le n-1\} \quad \cup \\ \{y_{n-j-k,j}^{p^k} : 0 \le j \le n-1, 0 \le j+k \le n-1\} \quad \cup \\ \{u_k^{p^{n+1-k}} : 0 \le k \le n\} \end{gathered}$$

As in the proof of 4.17,we can use the Hopf map H to rule out many differentials. If some $x_{n+1-j,j}$ supports a nontrivial differential, its target must be in the kernel of H since $H(x_{n+1-j,j}) = x_{n-j,j}$ is a permanent cycle. Thus the target must be a multiple of either $y_{1,n-j}^{p^j}$ for $0 \le j \le n-1$ or $u_0^{p^{n+1}}$. Hence the expected differentials

$$d_{2p^j-1}(x_{n+1-j,j}) = v_n y_{1,n-1-j}^{p^j}$$

follow by induction on j.

In larger degrees we can rule out other differentials on $x_{n+i,j}$ as in the proof of 4.17.

In the proof of 4.17 we knew that each $y_{i,j}$ with $j \leq n-2$ was a permanent cycle because the remaining primitives in the same Snaith degree were also even dimensional. However, this argument is not good enough here because we do not know (and we will see that it is not true) that $Y(n)_*$ is even dimensional. We have to consider the possibility that

$$d_{p^{i+j+1-k}-2}(y_{i,j}) = \alpha u_k^{p^{i+j+1-k}} \tag{4.24}$$

for some $\alpha \in Y(n)_*$ and $n \leq k \leq 2n$.

We can use the Hopf map to exclude such a differential in the following way. The Snaith summands of $\Omega^3 S^{2dp+1}$ are independent of d. If d is divisible by p^m then our spectral sequence is in the image of the mth iterate of the Hopf map H. Thus we get a diagram

$$\begin{array}{ccc} y_{i,j} & \xrightarrow{d_r} & \alpha u_k^{p^{i+j+1-k}} \\ {\scriptstyle H^m}\uparrow & & \uparrow \\ y_{i+m,j} & \xrightarrow{d_{r'}} & \alpha' u_{k'}^{p^{i+j+m+1-k'}} \end{array} \tag{4.25}$$

where

$$\begin{aligned} r+2 &= p^{i+j+1-k} \\ \text{and} \qquad r'+2 &= p^{i+j+m+1-k'} \end{aligned}$$

with $n \leq, k, k' \leq 2n$. Since H^m induces a map of spectral sequences, we have $r \geq r'$, so

$$\begin{aligned} i+j+1-k &\geq i+j+m+1-k' \\ k' &\geq k+m \end{aligned}$$

This is incompatible with the upper bound on k' since we can do this for any value of m, so there can be no nontrivial differential of the form (4.24). □

We will now explain why the isomorphism of 4.22 need not be multiplicative. The argument given in the proof of 4.17 to show that there are no multiplicative extensions does *not* carry over to the computation of $Y(n)_*(\Omega^3 S^{1+2p^n})$. Indeed $y_{i,j}^{p^{n-1-j}}$ could be a nontrivial $Y(n)_*$-linear combination of the elements $u_k^{p^{n+i-k}}$ for $0 \leq k \leq n$, e.g.,

$$v_n^{-1} b_{i+j+1,n-2-j} u_0^{p^{n+i}}$$

when $i+j \geq n$. Here $b_{i,j}$ denotes the image of $y_{i-n,j}$ under the map of (3.18).

The use of the Hopf map in the last paragraph of the proof above is similar to its use by the first author in [Mah77], and it deserves further comment. In that paper there was a 2-local stable splitting

$$\Omega^2 S^9 \simeq \bigvee_{i>0} \Sigma^{7i} B_i$$

where the stable summand B_i is known now to be the Brown-Gitler spectrum $B([i/2])$ with bottom cell in dimension 0 and top cell in dimension $i-\alpha(i)$. There one wanted to show that a certain element

$$x_j \in \operatorname{Ext}_{A_*}^{1,1+2^j}(\mathbf{Z}/(2), H_*(B_{2^j}))$$

was a permanent cycle. The Hopf map induces

$$\Sigma^{-2^{j+1}} B_{2^{j+1}} \xrightarrow{H} \Sigma^{-2^j} B_{2^j}$$

which sends x_{j+1} to x_j. Now suppose there is a nontrivial differential

$$d_r(x_j) = z_j \in \operatorname{Ext}_{A_*}^{1+r,r+2^j}(\mathbf{Z}/(2), H_*(B_{2^j})).$$

It was shown that no such z_j is in the image of the Hopf map H, so the differential cannot occur.

This amounts to saying that there is an element

$$x \in \pi_0(\lim_{\leftarrow} \Sigma^{-2^j} B_{2^j})$$

which projects to an element representing x_j. Using properties of Brown-Gitler spectra one can produce maps

$$S^0 \to \Sigma RP_{-\infty}^{-2} \to \lim_{\leftarrow} \Sigma^{-2^j} B_{2^j}$$

where the first map is Spanier-Whitehead dual to the transfer map $t : RP_1^\infty \to S^0$, and the composite is the desired x.

Dually one can look for an element

$$x^* \in \pi^0(\lim_{\rightarrow} \Sigma^{2^j} DB_{2^j})$$

which is given by the composite

$$\lim_{\rightarrow} \Sigma^{2^j} DB_{2^j} \longrightarrow RP_1^\infty \xrightarrow{t} S^0.$$

In [Car83] Carlsson showed that $H^*(\lim_{\rightarrow} \Sigma^{2^j} DB_{2^j})$ has $H^*(RP_1^\infty)$ as a direct summand as an A-module, and that both are unstable injectives. Using results of Goerss-Lannes [GL87] or Lannes-Schwartz [LS89], one can deduce that the spectrum $\lim_{\rightarrow} \Sigma^{2^j} DB_{2^j}$ has RP_1^∞ as a retract.

One can ask analogous questions about the triple loop summands T_i. The Hopf map induces

$$\Sigma^{-2pi}T_{pi} \xrightarrow{H} \Sigma^{-2i}T_i$$

for each i. This map sends $y_{i+1,j}$ to $y_{i,j}$. Our proof shows that for each $0 \leq j \leq n-2$ there is an element

$$y_j \in \lim_{\leftarrow} Y(n)_{-2p^{j+1}-2}(\Sigma^{-2p^k}T_{p^k})$$

which projects to $y_{i,j}$ for each i. Note that $\lim_{\leftarrow} Y(n)_*(\Sigma^{-2p^k}T_{p^k})$ need not be the same as $Y(n)_*(\lim_{\leftarrow} \Sigma^{-2p^k}T_{p^k})$ since homology does not commute with inverse limits, but it is the former group which interests us here.

The limit problem disappears when we dualize, since homology and cohomology do commute with direct limits. We have an element

$$y_j^* \in Y(n)^{2p^{j+1}+2}(\lim_{\rightarrow} \Sigma^{2p^k}DT_{p^k}).$$

and similarly for Morava K-theory.

An analog of Carlsson's theorem would be the following.

Conjecture 4.26. *The spectrum* $\lim_{\rightarrow} \Sigma^{2p^k}DT_{p^k}$ *has a copy of the suspension spectrum of the Eilenberg-Mac Lane space* $K(\mathbf{Z}/(p), 2)$ *as a retract.*

Kuhn [Kuh] has recently proved the corresponding statement in cohomology for $p = 2$.

A proof of 4.26 (or the construction of a suitable map from the direct limit to $K(\mathbf{Z}/(p), 2)$) might lead to an independent construction of the elements y_j^* as follows. $K(n)^*(K(\mathbf{Z}/(p), 2))$ is known [RW80] and it is likely that $Y(n)^*(K(\mathbf{Z}/(p), 2))$ has a similar description. The former has $n-1$ algebra generators which might map to the desired y_j^*.

For more details, see [Rav98]

5. Toward a proof of the differentials conjecture

5.1. **The E_2-term of the localized Thomified Eilenberg-Moore spectral sequence.** Now we are ready to describe our program to prove our conjecture about differentials, 3.16. We will use the map of (3.18) from the localized Thomified Eilenberg-Moore spectral sequence for $Y(n)_*(\Omega^3 S^{1+2p^n})$ to the localized Adams spectral sequence for $Y(n)$. Because the material in this section becomes rather technical at times, we will illustrate each main idea in the case $n = 2$, where the degree of complexity is quite manageable. To begin this extended example, we recall from Corollary 2.11 that the localized Adams-Novikov spectral sequence for $\pi_*(L_2y(2))$ has

$$E_2 = K(2)_*[v_3, v_4] \otimes E(h_{3,0}, h_{4,0}, h_{3,1}, h_{4,1}).$$

In this case, the localized Adams-Novikov spectral sequence collapses from this point, because the only possible differential ($d_3(h_{3,1}) = v_2^{-p^4} h_{3,0}h_{3,1}h_{4,0}h_{4,1}$?) is ruled out because $h_{3,1}$ is easily seen to be a permanent cycle in the unlocalized Adams-Novikov spectral sequence for $\pi_*(y(2))$.

For all $n \geq 1$, we know from §3.3 that the E_2-term of the localized Thomified Eilenberg-Moore spectral sequence is

$$v_n^{-1}\mathrm{Ext}_{H_*(\Omega^2 S^{1+2p^n})\otimes B(n)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)),$$

where $B(n)_*$ is as in (2.2) and $H_*(\Omega^2 S^{1+2p^n})$ is given in (4.11).

Lemma 5.1. (i) *The E_2-term of the localized Thomified Eilenberg-Moore spectral sequence for $Y(n)_*(\Omega^3 S^{1+2p^n})$ is*

$$\begin{aligned}\mathrm{Ext}_{P'}\big(\mathbf{Z}/(p), v_n^{-1}\mathrm{Ext}_{Q\otimes H_*(\Omega^2 S^{1+2p^n})}(\mathbf{Z}/(p), \mathbf{Z}/(p))\big) \\ = R(n)_* \otimes E(h_{n+i,j}) \otimes P(b_{n+i,j}) \\ \otimes P(u_0, \ldots, u_n) \otimes E(\tilde{x}_{i,j}) \otimes P(\tilde{y}_{i,j}).\end{aligned}$$

where the indices i and j satisfy $i > 0$ and $0 \leq j \leq n-1$, and

$$R(n)_* = K(n)_*[v_{v+1}, \ldots, v_{2n}].$$

The elements $\tilde{x}_{i,j}$ and $\tilde{y}_{i,j}$ are related to the homology classes $x_{i,j}$ and $y_{i,j}$ in $H_*(\Omega^3 S^{1+2p^n})$ and will be defined below in (5.9) and (5.10).

(ii) *Under the map of (3.18),*

$$\begin{aligned} v_{n+k} &\mapsto v_{n+k}, \\ h_{n+i,j} &\mapsto h_{n+i,j}, \\ b_{n+i,j} &\mapsto b_{n+i,j}, \\ u_k &\mapsto v_{n+k}, \\ \tilde{x}_{i,j} &\mapsto h_{n+i,j}, \\ \textit{and} \quad \tilde{y}_{i,j} &\mapsto b_{n+i,j}. \end{aligned}$$

(iii) *The $H_*(y(n))$-comodule structure (as in Lemma 3.17) on these generators is given by*

$$\begin{aligned}
\psi(v_{n+i}) &= \sum_{0\le k\le i} \xi_{i-k}^{p^{n+k}} \otimes v_{n+k},\\
\psi(h_{n+i,j}) &= \sum_{0\le k\le n-1-j} \overline{\xi}_k^{p^j} \otimes h_{n+i-k,j+k},\\
\psi(b_{n+i,j}) &= \sum_{0\le k\le n-1-j} \overline{\xi}_k^{p^{j+1}} \otimes b_{n+i-k,j+k},\\
\psi(u_i) &= 1\otimes u_i + \sum_{0\le k\le i} \overline{\tau}_k \otimes \tilde{x}_{i-k,k},\\
\psi(\tilde{x}_{i,j}) &= \sum_{0\le k\le n-1-j} \overline{\xi}_k^{p^j} \otimes \tilde{x}_{i-k,j+k}\\
\textit{and} \qquad \psi(\tilde{y}_{i,j}) &= \sum_{0\le k\le n-1-j} \overline{\xi}_k^{p^{j+1}} \otimes \tilde{y}_{i-k,j+k}.
\end{aligned}$$

In the $n=2$ case, the localized Thomified Eilenberg-Moore spectral sequence has

$$\begin{aligned}
E_2 = K(2)_*[v_3, v_4]\otimes \quad & E(h_{3,0}, h_{4,0}, \dots, h_{3,1}, h_{4,1}, \dots)\\
&\otimes P(b_{3,0}, b_{4,0}, \dots, b_{3,1}, b_{4,1}, \dots)\\
\otimes P(u_0, u_1, u_2) \otimes & E(\tilde{x}_{1,0}, \tilde{x}_{2,0}, \dots, \tilde{x}_{1,1}, \tilde{x}_{2,1} \dots)\\
&\otimes P(\tilde{y}_{1,0}, \tilde{y}_{2,0}, \dots, \tilde{y}_{1,1}, \tilde{y}_{2,1} \dots).
\end{aligned}$$

The values of the map of 3.18 and the coaction are easily read off from the Lemma above.

Proof. (i) We begin by describing the coproduct in $H_*(\Omega^2 S^{1+2p^n}) \otimes B(n)_*$ using (2.24) and (4.12). Elements in $1\otimes B(n)_*$ have their usual coproduct, while coproducts for the generators of $H_*(\Omega^2 S^{1+2p^n})\otimes 1$ are given by

$$\begin{aligned}
e_i\otimes 1 &\mapsto e_i\otimes 1\otimes 1\otimes 1 + 1\otimes 1\otimes e_i\otimes 1\\
&\qquad + \sum_{0\le k< i-n} 1\otimes \overline{\tau}_{n+k}\otimes f_{i-n-k}^{p^{n+k}}\otimes 1\\
\text{and} \qquad f_i\otimes 1 &\mapsto f_i\otimes 1\otimes 1\otimes 1 + 1\otimes 1\otimes f_i\otimes 1\\
&\qquad + \sum_{0< k< i-n} 1\otimes \overline{\xi}_{n+k}\otimes f_{i-n-k}^{p^{n+k}}\otimes 1.
\end{aligned}$$

The best way to get at the cohomology of the semitensor product $H_*(\Omega^2 S^{1+2p^n}) \otimes B(n)_*$ is via the Hopf algebra extension

$$F \otimes P'_* \longrightarrow H_*(\Omega^2 S^{1+2p^n}) \otimes B(n)_* \longrightarrow E \otimes Q'_* \tag{5.2}$$

In the Cartan-Eilenberg spectral sequence for this, we have

$$\begin{aligned} E_2 &= \operatorname{Ext}_{F\otimes P'_*}(\mathbf{Z}/(p), \operatorname{Ext}_{E\otimes Q'_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))) \\ &= \operatorname{Ext}_{F\otimes P'_*}(\mathbf{Z}/(p), U \otimes V') \end{aligned}$$

where P'_*, Q'_* and V' are as in (2.4), (2.5), and (2.7),

$$\begin{aligned} U &= P(u_0, u_1, \ldots), \\ E &= E(e_0, e_1, \ldots), \\ \text{and} \quad F &= \begin{cases} P(f_1, f_2, \ldots) & \text{for } p > 2 \\ P(e_0^2, e_1^2, \ldots) & \text{for } p = 2. \end{cases} \end{aligned}$$

The coalgebra structure of $F \otimes P'_*$ and the structure of $U \otimes V'$ as a left comodule over it for odd primes are given by

$$\begin{cases} \xi_i \mapsto \sum_{0\le k\le i} \xi_{i-k}^{p^k} \otimes \xi_k, \\ f_i \mapsto f_i \otimes 1 + \sum_{0\le k<i} \overline{\xi}_k \otimes f_{i-k}^{p^k}, \\ v_i \mapsto \sum_{0\le k\le i} \xi_{i-k}^{p^k} \otimes v_k \\ \text{and} \quad u_i \mapsto 1 \otimes u_i + \sum_{0\le k<i} \overline{f}_{i-k}^{p^k} \otimes v_k, \end{cases} \tag{5.3}$$

where $\xi_0 = 1$, $\xi_i = 0$ for $0 < i \le n$, $v_i = 0$ for $0 \le i < n$, $\overline{\xi}_k$ is the conjugate of ξ_k, and $\overline{f}_k$ is the conjugate of f_k in $F \otimes P'_*$. These formulas follow from the coproduct in A_* and the coaction of A_* on $H_*(\Omega^2 S^{1+2p^n})$.

As in (2.16) we can enlarge $U \otimes V'$ to a ring

$$Z = P(u_0, ..., u_n, z_{n+1}, \ldots; v_n, ..., v_{2n}, w_{2n+1}, \ldots) \subset v_n^{-1} U \otimes V'; \tag{5.4}$$

where w_{2n+i} is as in (2.15), and z_{n+i} is defined recursively for $i > 0$ by a similar formula,

$$z_{n+i} = v_n^{-1} \left(u_{n+i} - \sum_{0<k<i} v_{n+k} z_{n+i-k}^{p^k} \right). \tag{5.5}$$

Note that

$$v_n^{(p^i-1)/(p-1)} z_{n+i} \in U \otimes V'$$

and that

$$v_n^{(p^i-1)/(p-1)} z_{n+i} \equiv (-1)^{i+1} v_{n+1}^{(p^{i-1}-1)/(p-1)} u_{n+1}^{p^{i-1}} \bmod (v_n). \tag{5.6}$$

We claim that

$$\psi(z_{n+i}) = 1\otimes z_{n+i} - f_i^{p^n}\otimes 1 + \sum_{n<k<i} \overline{\xi}_k^{p^n}\otimes z_{n+i-k}^{p^k}. \tag{5.7}$$

The formula for $\psi(w_{2n+i})$ of (2.18) is the homomorphic image of this under the map of (3.18). We will verify the claim after proving the rest of the lemma.

It follows from (5.7) that Z is free as a comodule over

$$P(f_1^{p^n}, f_2^{p^n}, \ldots)\otimes P(\xi_{n+1}^{p^n}, \xi_{n+2}^{p^n}, \ldots)$$

and that

(5.8)

$$\begin{aligned}
&\mathrm{Ext}_{F\otimes P'_*}(\mathbf{Z}/(p), Z)\\
&\quad = P(u_0, \ldots u_n)\otimes P(v_n, \ldots, v_{2n})\otimes\\
&\qquad\qquad \mathrm{Ext}_{F/(f_i^{p^n})\otimes P'_*/(\xi_{n+i}^{p^n}))}(\mathbf{Z}/(p), \mathbf{Z}/(p))\\
&\quad = P(u_0, \ldots u_n)\otimes P(v_n, \ldots, v_{2n})\\
&\qquad\qquad \otimes\begin{cases} E(\tilde{x}_{i,j})\otimes P(\tilde{y}_{i,j}) & \\ \qquad \otimes E(h_{n+i,j})\otimes P(b_{n+i,j}) & \text{for } p \text{ odd}\\ P(\tilde{x}_{i,j})\otimes P(h_{n+i,j}) & \text{for } p=2,\end{cases}
\end{aligned}$$

where the indices i and j satisfy $i>0$ and $0\le j\le n-1$, $\tilde{x}_{i,j}$ corresponds (roughly speaking; see (5.9) below for the precise definition) to the element $-f_i^{p^j}$ (which is primitive in the quotient $F/(f_i^{p^n})\otimes P'_*/(\xi_{n+i}^{p^n})$), and $\tilde{y}_{i,j}$ is its transpotent. In particular this Ext group is v_n-torsion free.

As in (2.20), localizing inverts v_n, and we get the stated value of E_2 for the Cartan-Eilenberg spectral sequence for (5.2). To show that the spectral sequence collapses from E_2, it suffices to show that the elements u_k and v_{n+k} for $0\le k\le n$ are permanent cycles. This follows from the fact that e_k and τ_{n+k} are primitive for these k.

Next we need to define the elements $\tilde{x}_{i,j}$ and $\tilde{y}_{i,j}$. The localized double complex associated with (5.2) is

$$C_{F\otimes P'_*}(v_n^{-1}C_{E\otimes Q'_*}(\mathbf{Z}/(p))).$$

The algebra $E\otimes Q'_*$ is primitively generated, so we can replace its localized cobar complex by its localized Ext group $U\otimes v_n^{-1}V'$. The coaction of $F\otimes P'_*$ on it was described above in (5.3), and the corresponding local Ext group was given in (5.8). In the cobar complex we have by

(5.7) and (2.28),

$$d(z_{n+i}) = -f_i^{p^n} \otimes 1 + \sum_{n<k<i} \overline{\xi}_k^{p^n} \otimes z_{n+i-k}^{p^k},$$

so this element is a cocycle. It follows that

$$\tilde{x}_{i,j} = -f_i^{p^j} \otimes 1 + \sum_{n<k<i} \overline{\xi}_k^{p^j} \otimes z_{n+i-k}^{p^{j+k-n}} \tag{5.9}$$

is also one.

To define $\tilde{y}_{i,j}$ (the transpotent of $\tilde{x}_{i,j}$), we proceed as follows. We regard $\tilde{x}_{i,j}$ as an element in the algebra $F \otimes P'_* \otimes U \otimes v_n^{-1}V'$ and let $\tilde{x}_{i,j}^{(m)}$ denote the image of its mth power in the algebra

$$C^1_{F\otimes P'_*}(U \otimes v_n^{-1}V') = F \otimes P'_* \otimes U \otimes v_n^{-1}V'.$$

Then a routine calculation (using only the fact that $x_{i,j}$ is a cocycle) shows that

$$d(\tilde{x}_{i,j}^{(m)}) = \sum_{0<\ell<m} \binom{m}{\ell} \tilde{x}_{i,j}^{(\ell)} \cup \tilde{x}_{i,j}^{(m-\ell)},$$

where $\cup$ denotes the cup product partly described in (2.29). It follows that

(5.10)

$$\begin{aligned} \tilde{y}_{i,j} &= \textstyle\sum_{0<m<p} p^{-1}\binom{p}{m} \tilde{x}_{i,j}^{(m)} \cup \tilde{x}_{i,j}^{(p-m)} \\ &= -\textstyle\sum_{0<m<p} p^{-1}\binom{p}{m} f_i^{p^j m} \otimes f_i^{p^j(p-m)} \otimes 1 \\ &\quad + \textstyle\sum_{n<k<i}\sum_{0<m<p} p^{-1}\binom{p}{m} \overline{\xi}_k^{p^{j+k-n}m} \otimes \overline{\xi}_k^{p^{j+k-n}(p-m)} \otimes z_{n+i-k}^{p^{1+j+k-n}} + \dots \end{aligned}$$

is the desired cocycle.

(ii) The assertion about the map of (3.18) follows immediately from the definitions of the elements in question; compare (5.9) with (2.19).

(iii) The first three cases were proved in Lemma 3.17. The last three can be read off from the A_*-comodule structure on $H_*(\Omega^3 S^{1+2p^n})$.

Finally, we need to verify (5.7). We will do so by manipulating the following power series in a dummy variable t. Let

$$
\begin{array}{rcll}
v(t) & = & \sum_{i\geq 0} v_{n+i}t^{p^i}, & \\
\hat{v}(t) & = & \sum_{i>0} v_{2n+i}t^{p^{n+i}}, & \\
\overline{v}(t) & = & v^{-1}(t), & \\
 & & \text{the functional inverse of } v(t), & \\
u(t) & = & \sum_{i\geq 0} u_i t^{p^i}, & \\
\hat{u}(t) & = & \sum_{i>0} u_{n+i}t^{p^{n+i}}, & \\
w(t) & = & \sum_{i>0} w_{2n+i}t^{p^{n+i}}, & \\
z(t) & = & \sum_{i>0} z_{n+i}t^{p^{n+i}}, & \\
\xi(t) & = & \sum_{i\geq 0} \xi_i^{p^n} t^{p^i}, & \\
\overline{\xi}(t) & = & \sum_{i\geq 0} \overline{\xi}_i^{p^n} t^{p^i}, & \\
f(t) & = & \sum_{i\geq 0} f_i^{p^n} t^{p^{n+i}} & \text{where } f_0 = 1, \\
\text{and } \overline{f}(t) & = & \sum_{i\geq 0} \overline{f}_i^{p^n} t^{p^{n+i}}. &
\end{array}
\tag{5.11}
$$

In what follows we will drop the variable t and denote functional composition by the symbol $\circ$. In this way (5.3) can be rewritten as

$$
\begin{array}{lrcl}
 & \psi(v) & = & (1\otimes v)\circ(\xi\otimes 1) \\
\text{so} & \psi(\overline{v}) & = & (\overline{\xi}\otimes 1)\circ(1\otimes\overline{v}), \\
 & \psi(\hat{u}) & = & 1\otimes\hat{u} + (1\otimes v)\circ((\overline{f}-1)\otimes 1), \\
\text{and} & \psi(f-1) & = & (\overline{\xi}\otimes 1)\circ(1\otimes(f-1)) + (f-1)\otimes 1 \\
\text{so} & \overline{\xi}\circ\overline{f} & = & 1+\overline{\xi}-f
\end{array}
$$

since in any connected Hopf algebra, $\psi(x) = x'\otimes x''$ implies that $x'\bar{x}'' = 0$.

Similarly (2.15) and (5.5) can be rewritten as

$$
\begin{array}{lrcl}
 & \psi(\hat{u}) & = & 1\otimes\hat{u} + (1\otimes v)\circ((\overline{f}-1)\otimes 1), \\
 & \hat{u} & = & v\circ z \\
\text{so} & z & = & \overline{v}\circ\hat{u}, \\
\text{and} & \hat{v} & = & v\circ w \\
\text{so} & w & = & \overline{v}\circ\hat{v}.
\end{array}
$$

Thus we have

$$\begin{aligned}\psi(z) &= \psi(\overline{v})\circ\psi(\hat{u})\\ &= (\overline{\xi}\otimes 1)\circ(1\otimes\overline{v})\circ(1\otimes\hat{u}+(1\otimes v)\circ((\overline{f}-1)\otimes 1))\\ &= (\overline{\xi}\otimes 1)\circ(1\otimes\overline{v}\circ\hat{u})+(\overline{\xi}\otimes 1)\circ((\overline{f}-1)\otimes 1)\\ &= (\overline{\xi}\otimes 1)\circ(1\otimes z)+\overline{\xi}\circ(\overline{f}-1)\otimes 1\\ &= (\overline{\xi}\otimes 1)\circ(1\otimes z)+(1-f)\otimes 1,\end{aligned}$$

which is a reformulation of (5.7). □

5.2. **Short differentials.** It follows from 5.1(ii) that a differential on $\tilde{x}_{i,j}$ forces a similar one on $h_{n+i,j}$, and if $\tilde{y}_{i,j}$ is a permanent cycle so is $b_{n+i,j}$. As in the computations of §4.3 and §4.4 we have more control over such differentials because they must respect the Snaith splitting and the Hopf algebra structure. Thus Conjecture 3.14 is a consequence of the following.

Conjecture 5.12. *For $i>0$ and $0\le j\le n-1$, the element $\tilde{x}_{n+i-j,j}$ survives to E_{2p^j} and supports a nontrivial differential*

$$d_{2p^j}(\tilde{x}_{n+i-j,j}) = v_n\tilde{y}^{p^j}_{i,n-1-j}.$$

Each $\tilde{y}_{i,j}$ for $i>0$ and $0\le j\le n-2$ survives to $E_{1+2p^{n-1}}$.

Note that if in addition each $\tilde{y}_{i,j}$ were a permanent cycle, then we would have

$$\begin{aligned}E_\infty &= R(n)_*\otimes P(u_0,\dots,u_n)\otimes E(\tilde{x}_{i,j},h_{n+i,j}\colon i+j\le n)\\ &\otimes\ P(\tilde{y}_{i,j},b_{n+i,j})/(\tilde{y}^{p^{n-1-j}}_{i,j},b^{p^{n-1-j}}_{n+i,j}).\end{aligned} \tag{5.13}$$

In the $n=2$ case, Conjecture 5.12 predicts that all the $\tilde{x}_{i,0}$s survive to E_2 and that $d_2(\tilde{x}_{3,0})=v_2\tilde{y}_{1,1}$, $d_2(\tilde{x}_{4,0})=v_2\tilde{y}_{2,1}$,, etc. Further, we expect that all the $\tilde{x}_{i,1}$s survive to E_{2p}, and that $d_{2p}(\tilde{x}_{2,1})=v_2\tilde{y}^p_{1,0}$, $d_{2p}(\tilde{x}_{3,1})=v_2\tilde{y}^p_{2,0}$, etc. If the $\tilde{y}_{i,0}$s were all permanent cycles, the the localized Thomified Eilenberg-Moore spectral sequence would collapse from E_{2p} and have

$$\begin{aligned}E_\infty = K(2)_*[v_3,v_4]\ &\otimes P(u_0,u_1,u_2)\otimes E(h_{3,0},h_{4,0},h_{3,1},\tilde{x}_{1,0},\tilde{x}_{2,0},\tilde{x}_{1,1})\\ &\otimes P(b_{3,0},b_{4,0},\dots,\tilde{y}_{1,0},\tilde{y}_{2,0},\dots)/(b^p_{i,0},\tilde{y}^p_{j,0}).\end{aligned}$$

For all $n>0$, one might think that Conjecture 5.12 is a consequence of Theorems 4.17 and 4.22, but this is not the case. The differentials of 4.17 suggest but do not actually imply those of 5.12, because the two spectral sequences are based on different filtrations. The method used in the proof of 4.22 to show that the $y_{i,j}$ are permanent cycles does not imply that the $\tilde{y}_{i,j}$ are.

That method *can* be used to show that there is an element congruent to $\tilde{y}_{i,j}$ modulo decomposables, namely $H^{n+1}(\tilde{y}_{i+n+1,j})$ (the image of $\tilde{y}_{i+n+1,j}$ under the $(n+1)$th iterate of the Hopf map H), which is a permanent cycle. We can use Lemma 5.16 below to identify it as

$$H^{n+1}(\tilde{y}_{i+n+1,j}) = \tilde{y}_{i,j} - \sum_{0\le k\le n} \beta_{i+k,j}(v_n^{-1}u_{n-k})^{p^{i+j+k+1-n}} \tag{5.14}$$

for certain coefficients $\beta_{i+k,j}$ defined in 5.16. The image of this element under the map of (3.18) is

$$b_{n+i,j} - \sum_{0\le k\le n} \beta_{i+k,j}(v_n^{-1}v_{2n-k})^{p^{i+j+k+1-n}} = 0 \qquad \text{by 5.16 below.}$$

Previously we had thought this image was congruent to $b_{n+i,j}$ modulo decomposables, which would imply the collapsing of the localized Adams spectral sequence for $Y(n)$, but unfortunately this is not the case. Thus the survival of the element of (5.14) is of no help in determining the structure of $Y(n)_*$. On the other hand, the low dimensional computation showing that $\tilde{x}_{i,j}$ survives does imply the survival of $h_{n+i,j}$ for $i+j\le n$.

We have instead

Conjecture 5.15 (Second differentials conjecture)**.** *In the localized Adams spectral sequence for $Y(n)$ for $n>1$ the elements $h_{n+i,0}$ and $h_{n+i,1}$ survive to E_2 and E_{2p} respectively, and there are differentials*

$$\begin{aligned} d_2(\tilde{x}_{n+i,0}+s_{n+i,0}) &= v_n\tilde{y}_{i,n-1} \\ \text{and} \qquad d_{2p}(\tilde{x}_{n+i-1,1}+s_{n+i-1,1}) &= v_n\tilde{y}_{i,n-2}^p \end{aligned}$$

for decomposables $s_{n+i-j,j}$ (not to be confused with, but mapping to the decomposables $s_{2n+i-j,j}$ of Theorem 3.16). The elements $\tilde{y}_{i,j}$ for $j<n-1$ survive to E_{2p+1}, so

$$\begin{aligned} E_{2p+1} = {} & R(n)_*\otimes P(u_0,\ldots,u_n)\otimes E(\tilde{x}_{i,0},h_{n+i,0}: 1\le i\le n) \\ & \otimes E(\tilde{x}_{i,1},h_{n+i,1}: 0\le i\le n-1) \\ & \otimes E(\tilde{x}_{i,j},h_{n+i,j}: i>0, 2\le j\le n-1) \\ & \otimes P(\tilde{y}_{i,n-2},b_{n+i,n-2}: i>0)/(\tilde{y}_{i,n-2}^p,b_{n+i,n-2}^p) \\ & \otimes P(\tilde{y}_{i,j},b_{n+i,j}: i>0, 0\le j\le n-3). \end{aligned}$$

We will make use of the Hopf map as before. Consider the following diagram in which both rows are fiber sequences.

$$\begin{array}{ccccc}
\Omega J_{p^n-1}S^2 \times \Omega^3 S^{1+2p^n} & \longrightarrow & \Omega^2 S^3 \times \text{pt.} & \longrightarrow & \Omega^2 S^{1+2p^n} \times \Omega^2 S^{1+2p^n} \\
\downarrow {\scriptstyle \Omega J_{p^n-1}S^2 \times H^m} & & \| & & \downarrow {\scriptstyle \Omega^2 S^{1+2p^n} \times H^m} \\
\Omega J_{p^n-1}S^2 \times \Omega^3 S^{1+2p^{m+n}} & \longrightarrow & \Omega^2 S^3 \times \text{pt.} & \longrightarrow & \Omega^2 S^{1+2p^n} \times \Omega^2 S^{1+2p^{m+n}}
\end{array}$$

This gives us a map from the Thomified Eilenberg-Moore spectral sequence for $y(n)_*(\Omega^3 S^{1+2p^n})$ to the one for $y(n)_*(\Omega^3 S^{1+2p^{n+k}})$. Now the second triple loop space has the same Snaith summands as the first, so the two spectral sequences are isomorphic. Thus the Hopf map H induces an endomorphism of our spectral sequence which is multiplicative and linear over $R(n)_* \otimes E(h_{n+i,j}) \otimes P(b_{n+i,j})$.

Lemma 5.16. *The Hopf map described above sends*

$$\begin{array}{rrcl}
 & u_{k+1} & \mapsto & u_k, \\
 & \tilde{x}_{i+1,j} & \mapsto & \tilde{x}_{i,j} - \eta_{i,j}(v_n^{-1}u_n)^{p^{i+j-n}} \\
\textit{and} & \tilde{y}_{i+1,j} & \mapsto & \tilde{y}_{i,j} - \beta_{i,j}(v_n^{-1}u_n)^{p^{i+j+1-n}}.
\end{array}$$

where the coefficients $\eta_{i,j}$ and $\beta_{i,j}$ vanish for $i \leq n$ and are defined recursively by

$$\begin{array}{rrcl}
 & h_{n+i,j} & = & \sum_{0\leq k\leq n} \eta_{i+k,j}(v_n^{-1}v_{2n-k})^{p^{i+j+k-n}} \\
 & & \equiv & \eta_{n+i,j} \bmod (v_{n+1}, \ldots, v_{2n}) \\
\textit{and} & b_{n+i,j} & = & \sum_{0\leq k\leq n} \beta_{i+k,j}(v_n^{-1}v_{2n-k})^{p^{i+j+k+1-n}} \\
 & & \equiv & \beta_{n+i,j} \bmod (v_{n+1}, \ldots, v_{2n}).
\end{array}$$

This along with Conjecture 3.16 implies that

$$d_{2p^j}(\eta_{n+i-j,j}) = v_n \beta_{i,n-1-j}^{p^j} \tag{5.17}$$

for $j = 0, 1$.

Proof. The value on u_{k+1} is immediate.

To evaluate $H(\tilde{x}_{i+1,j})$ we need first to compute $H(z_{n+i+1})$ for z_{n+i+1} as defined in (5.5). Let

$$r_i = \begin{cases} 0 & \text{for } i = 0 \\ v_n^{-1}\left(v_{n+i} - \sum_{0<k<i} v_{n+k} r_{i-k}^{p^k}\right) & \text{for } i > 0. \end{cases}$$

We will show by induction on i that

$$H(z_{n+i+1}) = \begin{cases} v_n^{-1}u_n & \text{for } i=0 \\ z_{n+i} - (v_n^{-1}u_n)^{p^i}r_i & \text{for } i>0. \end{cases}$$

This is immediate for $i=0$. For the inductive step with $i>0$, write

$$z_{n+i+1} = v_n^{-1}\left(u_{n+i+1} - \sum_{0<k\le i} v_{n+k}z_{n+i+1-k}^{p^k}\right),$$

so we have

$$\begin{aligned} H(z_{n+i+1}) &= v_n^{-1}\left(H(u_{n+i+1}) - \sum_{0<k\le i} v_{n+k}H(z_{n+i+1-k}^{p^k})\right) \\ &= v_n^{-1}\left(u_{n+i} - \sum_{0<k\le i} v_{n+k}(z_{n+i-k} - r_{i-k}(v_n^{-1}u_n)^{p^{i-k}})^{p^k}\right) \\ &= z_{n+i} - v_n^{-1}v_{n+i}(v_n^{-1}u_n)^{p^i} + \sum_{0<k\le i} v_{n+k}r_{i-k}^{p^k}(v_n^{-1}u_n)^{p^i} \\ &= z_{n+i} - r_i(v_n^{-1}u_n)^{p^i} \end{aligned}$$

as claimed.

Now the Hopf map commutes with the coboundary, so we have

$$\begin{aligned} H(\tilde{x}_{i+1,n}) &= H(d(z_{n+i+1})) \\ &= d(H(z_{n+i+1})) \\ &= d(z_{n+i} - w_{n+i}(v_n^{-1}u_n)^{p^i}) \\ &= \tilde{x}_{i,n} - d(w_{n+i})(v_n^{-1}u_n)^{p^i}. \end{aligned}$$

It follows that $H(\tilde{x}_{i,j})$ is as indicated where

$$\eta_{i,n} = d(w_{n+i})$$

and $\eta_{i,j}$ is its p^{j-n}th power. For $i \le n$ this vanishes since w_{n+i} is a cocycle.

For the recursive definition we use the series notation of (5.11) with

$$r(t) = \sum_{i>0} r_i t^{p^i},$$

so we have

$$v \circ r = v - v_n.$$

It follows that

$$\begin{aligned} r &= \overline{v} \circ (v - v_n) \\ &= 1 - \overline{v} \circ v_n \\ \text{so} \qquad r \circ v_n^{-1} \circ (v - \hat{v}) &= v_n^{-1} \circ (v - \hat{v}) - \overline{v} \circ (v - \hat{v}) \\ &= v_n^{-1} \circ (v - \hat{v}) - 1 + \overline{v} \circ \hat{v} \\ &= v_n^{-1} \circ (v - \hat{v}) - 1 + w \end{aligned}$$

Now the expression $v_n^{-1} \circ (v - \hat{v})$ is concentrated in dimensions below that of w_{2n+1}. Thus for each $i > 0$ we have

$$w_{2n+i} = \sum_{0 \le k \le n} r_{i+k} (v_n^{-1} v_{2n-k})^{p^{i+k}}.$$

Taking the coboundary gives

$$h_{n+i,n} = \sum_{0 \le k \le n} \eta_{i+k,n} (v_n^{-1} v_{2n-k})^{p^{i+k}}.$$

so

$$h_{n+i,j} = \sum_{0 \le k \le n} \eta_{i+k,j} (v_n^{-1} v_{2n-k})^{p^{i+j+k-n}}.$$

Taking the transpotent of the above gives the desired formula for $H(\tilde{y}_{i+1,j})$. □

The proof of 5.1(i) can be modified to give a similar description of the unlocalized E_2-term in Snaith degrees less than p^{n+1}. One can make an argument similar to that of 3.5 to show that there are no differentials in that range.

Alternately, one can look at the ordinary Adams spectral sequence for $y(n)_*(\Omega^3 S^{1+2p^n})$. Using the skeletal filtration one gets a prespectral sequence converging to the Adams E_2-term with

$$E_2 = H_*(\Omega^3 S^{1+2p^n}) \otimes \mathrm{Ext}_{B(n)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)).$$

Again there is no room for differentials in the range of the homology of the p^nth Snaith summand. The first differentials occur in Snaith degree p^{n+1}. They are induced by the Milnor operation Q_n given in (4.8), namely

(5.18)
$$\begin{aligned} d_{2p^n-1}(u_{n+1}) &= v_n x_{1,n} && \text{and} \\ d_{2p^n-1}(x_{n+1-j,j}) &= v_n y_{1,n-1-j}^{p^j} && \text{for } 0 \le j \le n-1, \end{aligned}$$

where differentials are indexed by the skeletal filtration. These give the differentials of Conjecture 5.15 for $i = 1$, and (via the map of (3.18)) those of (3.6).

This enables us to proceed by induction on Snaith degree, using the Hopf endomorphism described in Lemma 5.16. Assume inductively that the differentials on $\tilde{x}_{n+i-j-1,j}$ and $\eta_{n+i-j-1,j}$ are as stated in 3.16 and (5.17). Differentials must commute with the Hopf map, so *if* $\tilde{x}_{n+i-j,j}$ survives to E_{2p^j} we have

$$\begin{aligned} H(d_{2p^j}(\tilde{x}_{n+i-j,j})) &= d_{2p^j}(H(\tilde{x}_{n+i-j,j})) \\ &= d_{2p^j}(\tilde{x}_{n+i-j-1,j} - \eta_{n+i-j-1,j}(v_n^{-1}u_n)^{p^{i-1}})) \\ &= v_n y_{i-1,n-1-j}^{p^j} - v_n \beta_{i-1,n-1-j}^{p^j}(v_n^{-1}u_n)^{p^{i-1}} \\ &= H(v_n \tilde{y}_{i,n-1-j}^{p^j}). \end{aligned}$$

This means that if $\tilde{x}_{n+i-j,j}$ survives to E_{2p^j} then

$$d_{2p^j}(\tilde{x}_{n+i-j,j}) = v_n \tilde{y}_{i,n-1-j}^{p^j} + c_{i,j}, \tag{5.19}$$

where the error term $c_{i,j}$ must be a Hopf algebra primitive of Snaith degree p^{n+i} that is in the kernel of the Hopf endomorphism H.

The other possibility is that $\tilde{x}_{n+i-j,j}$ does not survive to E_{2p^j} but supports an earlier differential of the form

$$d_r(\tilde{x}_{n+i-j,j}) = c_{i,j} \qquad \text{for } 2 \le r < 2p^j, \tag{5.20}$$

where $c_{i,j}$ is as above. Similarly we can assume inductively that $\tilde{y}_{n+i-j,j-1}$ survives to E_{2p+1}, so a differential on $\tilde{y}_{n+i-j,j-1}$ must have the form

$$d_r(\tilde{y}_{n+i-j,j-1}) = c'_{i,j} \qquad \text{for } 2 \le r \le 2p, \tag{5.21}$$

where $c'_{i,j}$ has the same properties as $c_{i,j}$. We will refer to such unwanted differentials as *spurious* and show they cannot occur by showing that there are no nontrivial elements $c_{i,j}$ and $c'_{i,j}$ as above. We will use the structure of the triple loop space $\Omega^3 S^{1+2p^n}$.

5.3. **Excluding spurious differentials.** The fact that the error terms $c_{i,j}$ and $c'_{i,j}$ are primitives of Snaith degree p^{n+i} in the kernel of the Hopf endomorphism means that they must have the form

$$c_{i,j} = \gamma_{i,j} u_0^{p^{n+i}} + \sum_{0 \le k \le n-1} \alpha_{i,j,k} \tilde{y}_{1,k}^{p^{n+i-k-2}} \tag{5.22}$$

$$\text{and} \quad c'_{i,j} = \gamma'_{i,j} u_0^{p^{n+i}} + \sum_{0 \le k \le n-1} \alpha'_{i,j,k} \tilde{y}_{1,k}^{p^{n+i-k-2}} \tag{5.23}$$

where the coefficients $\gamma_{i,j}$,$\alpha_{i,j,k}$, $\gamma'_{i,j}$ and $\alpha'_{i,j,k}$ are in the E_r-term of the localized Adams spectral sequence for $Y(n)$ as follows.

$$\begin{array}{rcl} \gamma_{i,j} & \in & E_r^{1+r-p^{n+i},\,3p^{n+i}-2p^j+r-2} \\ \alpha_{i,j,k} & \in & E_r^{1+r-2p^{n+i-k-2},\,2p^{n+i-1}-2p^j+r-1} \\ \gamma'_{i,j} & \in & E_r^{2+r-p^{n+i},\,3p^{n+i}-2p^j+r-2} \\ \alpha'_{i,j,k} & \in & E_r^{2+r-2p^{n+i-k-2},\,2p^{n+i-1}-2p^j+r-1} \end{array}$$

Note that there is no hope of excluding these coefficients by simple sparseness arguments, because there are only finitely positive values of t for which the group $E_r^{s,\,s|v_n|+t}$ vanishes for small r.

These filtrations of these coefficients are negative, but the spurious differentials must lift back to the unlocalized Thomified Eilenberg-Moore spectral sequence, and there the coefficients must have nonnegative filtration. The element $\tilde{x}_{i,j}$ or $\tilde{y}_{i,j}$ need not be in the image of the unlocalized E_r-term, but some v_n-multiple of each must be until we get to the stage where it supports a localized differential. The power of v_n could increase with r if there is an unlocalized d_r with a target in the v_n-torsion.

Thus in order to exclude spurious differentials, we will proceed as follows.

(i) Find the smallest v_n-multiple of $\tilde{x}_{i,j}$ which is in the image of the unlocalized E_2-term.
(ii) Get an upper bound (depending on dimension) of the v_n-torsion in the unlocalized E_2-term.
(iii) Show that the torsion created in E_3 by the expected d_2s does not exceed this upper bound.
(iv) Use the torsion estimate to get information about smallest v_n-multiple of $\tilde{x}_{i,j}$ which is in the image of the unlocalized E_r-term. This will lead to restrictions on the coefficients in (5.22) and (5.23) which will enable us to exclude spurious differentials.

We do not know how to control the torsion in E_{2p+1} created by the d_{2p}s, and this difficulty prevents us from proving Conjectures 3.14 and 5.12 for $j>1$.

For step (i) above, let

$$e(i,j) = \begin{cases} 0 & \text{for } i \le 0 \\ \frac{p^{i+j}-p^j}{p-1} & \text{for } i > 1. \end{cases} \tag{5.24}$$

Then we can combine (5.6) with (5.9) to conclude that for $i>1$

$$v_n^{e(i-1,j+1)}\tilde{x}_{n+i,j} \equiv (-1)^i v_{n+1}^{e(i-2,j+1)}\overline{\xi}_{n+1}^{p^j} \otimes u_{n+1}^{p^{i+j-1}} \bmod (v_n). \tag{5.25}$$

This represents $(-1)^i v_{n+1}^{e(i-2,j+1)} h_{n+1,j} u_{n+1}^{p^{i+j-1}}$, which is nontrivial in the appropriate Ext group. Similarly one can show that $v_n^{e(i-1,j+2)} \tilde{y}_{n+i,j}$ has a nontrivial reduction modulo v_n.

For step (ii) above we have the following torsion estimate.

Lemma 5.26. *All v_n-torsion in the E_2 of the Thomified Eilenberg-Moore spectral sequence below dimension $2p^n(p^{n+i} + p^{n+1} - 2)$ is killed by $v_n^{e(i,0)}$.*

Proof. Recall from the proof of 5.1 that our E_2-term is isomorphic (up to regrading) to

$$\mathrm{Ext}_{F \otimes P'_*}(\mathbf{Z}/(p), U \otimes V').$$

Consider the short exact sequence of comodules over $F \otimes P'_*$,

$$0 \to U \otimes V' \to Z \to Z/(U \otimes V') \to 0,$$

where Z is an in (5.4). We know by (5.8) that the Ext group for Z is torsion free, so the torsion in E_2 all comes from the Ext group for the quotient comodule via the connecting homomorphism. It follows that the torsion in E_2 is controlled by that in the quotient itself. The first element there not killed by $v_n^{e(i,0)}$ is $z_{n+1} z_{n+i}$, which is in the indicated dimension. □

For (iii), the localized differential $d_2(\tilde{x}_{n+i,0}) = v_n \tilde{y}_{i,n-1}$ pulls back to

$$\begin{aligned} d_2(v_n^{e(i-1,1)} \tilde{x}_{n+i,0}) &= v_n^{e(i,0)} \tilde{y}_{i,n-1} \\ &= \begin{cases} v_n^{e(i,0)} \tilde{y}_{i,n-1} & \text{for } i \le n+1 \\ v_n^{e(n+1,0)} (v_n^{e(i-n-1,n+1)} y_{i,n-1}) & \text{for } i \ge n+2, \end{cases} \end{aligned}$$

so in E_3 the element $v_n^{e(i-n-1,n+1)} y_{i,n-1}$ for $i \ge n+2$, which is not divisible by v_n, has dimension

$$\frac{2p^n(p^{n+1}-1)(p^i-1)}{p-1} - 2,$$

and is killed by $v_n^{e(n+1,0)}$. This exponent does not exceed the one given by 5.26, so no higher torsion exists in E_3.

We now turn to step (iv). In (5.22) and (5.23) we can ignore the terms with $k = n-1$ since we know that $\tilde{y}_{1,n-1}$ is killed by d_2. The remaining coefficients with the largest filtrations are $\alpha_{i,j,n-2}$ and $\alpha'_{i,j,n-2}$ with

$$\begin{aligned} \mathrm{Filt}(\alpha_{i,j,n-2}) &= 1 + r - 2p^i \\ \mathrm{Filt}(\alpha'_{i,j,n-2}) &= 2 + r - 2p^i \end{aligned}$$

Thus in order to get a spurious value of $d_2(v_n^{e(i-1,1)}\tilde{x}_{n+i,0})$ we would need the quantity

$$3 + e(i-1,1) - 2p^i = (3-2p)e(i,0)$$

to be positive, but it never is. Thus $d_2(\tilde{x}_{n+i,0})$ is as claimed.

For the differential on $\tilde{x}_{n+i-1,1}$, we need to estimate the smallest v_n-multiple of it which is in the image of the unlocalized E_r-term. If we assume the worst, namely that at each stage there is a differential with a target having the largest order of v_n-torsion allowed by 5.26, namely $e(i,0)$. Then the filtration of this element is at most

$$1 + e(i-1,1) + (r-2)e(i,0) = (r-1)e(i,0). \tag{5.27}$$

It follows that for $r < 2p$ the filtration of $v_n^{e(i-1,1)+(r-2)e(i,0)}\alpha_{i,1,n-2}$ is at most $2p-3$. The product of any other coefficient with this power of v_n would have negative filtration, so the other coefficients must vanish.

Now we make use of the comodule structure of Lemma 5.1(iii). It implies that $\alpha_{i,1,n-2}$ cannot be divisible by v_{n+k} for $k > 0$. This means it suffices to consider it modulo $(v_{n+1}, \ldots, v_{2n})$. Its image there is a linear combination of elements of the form $v_n^e x$ with $e \geq 0$ and x in the subring generated by the surviving $h_{n+i,j}$ and $b_{n+i,j}$. The filtration of x must be a nonnegative multiple of $2p-2$, so it must be 0. This means our spurious differential has the form

$$d_r(\tilde{x}_{n+i-1,1}) = v_n^e \tilde{y}_{1,n-2}^{p^i}.$$

The exponent e is positive, which contradicts our assumption that $r < 2p$.

However we cannot exclude the case $r = 2p$, so in the localized Thomified Eilenberg-Moore spectral sequence for some $i \geq 3$ we could have

$$\begin{aligned} d_{2p}(\tilde{x}_{n+i-1,1}) &= v_n\tilde{y}_{n+i-2,0}^p + \alpha_{n-2}y_{1,n-2}^{p^i} \\ \text{so} \qquad d_{2p}(\tilde{x}_{n+i-1,1} - v_n^{-1}\alpha_{n-2}y_{1,n-2}^{p^i-p}\tilde{x}_{n,1}) &= v_n\tilde{y}_{n+i-2,0}^p. \end{aligned}$$

We still need to show that the elements $\tilde{y}_{n+i-j,j-1}$ for $0 < j < n$ and $\tilde{x}_{n+i-j,j}$ for $1 < j < n$ survive to E_{2p+1}. An argument similar to the one above shows that each survives to E_{2p}. At that stage we know that $\tilde{y}_{1,n-2}^p$ gets killed, so we can ignore the coefficients $\alpha_{i,j,n-2}$ and $\alpha'_{i,j,n-2}$, concentrating instead on $\alpha_{i,j,n-3}$ and $\alpha'_{i,j,n-3}$. Multiplying them by the worst power of v_n still gives an element with negative filtration, so we can exclude spurious d_{2p}s on these elements.

This completes the program to prove Conjectures 3.16 and 5.15.

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